VIBRATION OF RAIL AND ROAD VEHICLES

B. S. CAIN

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PREFACE

This book was originally conceived by the Committee on Industrial Aerodynamics of the American Society of Mechanical Engineers. Dr. Alexander Klemin, as Secretary of that Committee, invited the author to undertake the work, after discussion with Mr. H. L. Andrews.

The book has been written with the hope that it may help vehicle engineers and vibration specialists to appreciate each other's problems and points of view. For this reason the simple general principles are given whenever possible with only very elementary mathematics; but more complicated subjects, requiring more advanced mathematics, are also included.

The author is indebted to numerous sources for information and assistance, and wishes to express his thanks particularly to the following:

Dr. S. P. Timoshenko, whose books on vibration have been of great assistance; Prof. H. M. Jacklin, Dean A. A. Potter, of Purdue University, and the Society of Automotive Engineers for permission to reproduce diagrams and data relating to Human Reactions to Vibration; Prof. E. G. Keller and John Wiley and Sons, Inc., for permission to reproduce sections on Graeffe's method and on Determinants from *Mathematics of Modern Engineering* by Doherty and Keller; U. S. Rubber Products, Inc., whose pamphlet, *Some Physical Properties of Rubber*, has been an invaluable source of design information; Mr. R. K. Lee, whose S. A. E. speech of January 1932 was of great use in preparing the chapter on "Engine Mountings"; Mr. Robert N. Janeway and other engineers of the Chrysler Corporation for permission to reproduce illustrations of Floating Power; Dr. C. F. Hirshfeld and the Transit Research Corporation who have given invaluable assistance in preparing the section on Street Cars and who have allowed many records to be reproduced; Mr. Elmer Latshaw for valuable discussions of the riding qualities of cars and the J. G. Brill

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B. S. Cain

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VIBRATION OF RAIL AND ROAD VEHICLES

PART I. VIBRATION THEORY

Chapter 1

PRINCIPLES OF MECHANICS

Mechanical vibrations are subject to the same basic laws of mechanics which apply to all other motions. In many cases the basic laws can be applied directly to vibration problems and in all cases it is well to try to picture the conditions in terms of the fundamental principles. A few examples will be given here in order to recall those fundamental laws of mechanics which are most useful and to illustrate their direct application to some of the vibration problems which will be discussed later.

A body remains at rest or moves uniformly in a straight line unless acted upon by external force. An example of this basic law of motion is an internal combustion engine on a very soft flexible mounting, represented diagrammatically in Fig. 1. If the mounting is so soft that the engine floats freely, the engine as a whole will remain at rest because there is no force acting on it.

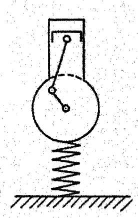


FIG. 1

Hence the center of gravity of the engine will not move. Therefore, as the piston moves up and down, the cylinder, crankcase, etc., must move down and up so that the center of gravity of the complete engine does not move. For example, if the piston and attached parts weigh 5 lb. and the remainder of the engine weighs 50 lb. and if the stroke is 6 in., the movement of the cylinders and crankcase will be

$$6 \text{ in.} \times \tfrac{5}{50} = 0.6 \text{ in.}$$

This applies regardless of the engine speed if the spring support is soft enough. For any spring support, if the engine speed is high enough, the internal forces will be very large compared with the spring force and the above calculation will be applicable.

The force acting on a body is equal to the mass of the body multiplied by its acceleration in the direction in which the force acts. This is the basis of most calculations made in later sections. At any moment,

$$\text{Force} = \text{mass} \times \text{acceleration.}$$

Over a period of time,

$$\text{Force} \times \text{time} = \text{mass} \times \text{acceleration} \times \text{time}$$

$$\text{or Force} \times \text{time} = \text{mass} \times \text{increase of velocity}$$

$$\text{or Force} \times \text{time} = \text{increase of momentum.}$$

Similar laws apply to angular motions. Returning to the simple engine in Fig. 1, the varying angular motion of the connecting rod produces an opposite angular oscillation of the remainder of the engine of an amount determined by the ratio of the moments of inertia. Also,

$$\text{Torque} = \text{moment of inertia} \times \text{angular acceleration.}$$

Conservation of Energy. The energy of a system is not changed unless work is done on the system, or the system does work or converts mechanical energy into heat energy through friction, etc.

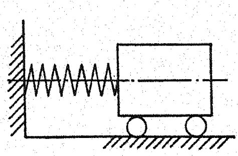


FIG. 2

For example, consider the weight supported on a spring, shown in Fig. 2. If the weight is pushed to compress the spring, say 1 in. with a total force of 15 lb., the average force required during compression is $7\frac{1}{2}$ lb. and the energy stored in the system is

$$1 \text{ in.} \times 7\tfrac{1}{2} \text{ lb.} = \tfrac{1}{12} \times 7\tfrac{1}{2} \text{ ft.-lb.} = 0.625 \text{ ft.-lb.}$$

If the weight is then released it will oscillate and the total energy will remain at 0.625 ft.-lb. unless decreased by friction. The total energy will be made up of the stored energy in the spring and the

kinetic energy of the oscillating weight. The maximum velocity of oscillation will occur when the kinetic energy is greatest; that is, when the stored energy in the spring is zero as the weight passes through its free position.

If the weight weighs, say, 48 lb., its kinetic energy is $\frac{48}{2 \times 32.2} V^2$ where V is its velocity in ft. per sec. Then when all the energy is kinetic,

$$\frac{48}{2 \times 32.2} V^2 = 0.625 \text{ ft.-lb.}$$

or $V = 0.92$ ft. per sec. is the maximum velocity during the oscillation.

When the energy of a system increases, work must be done on it and the work done is

Force $\times$ distance moved by the point of application of the force, in the direction of the force.

No oscillation can build up unless work is being done on it. Just as the energy input to a system is equal to the force multiplied by the distance through which it acts, so the rate of energy input or power input is equal to the force multiplied by the rate at which its point of application moves, or

Power = force $\times$ velocity of point of application of force.

Gyroscopic effects are merely examples of the laws of angular momentum, referred to above.

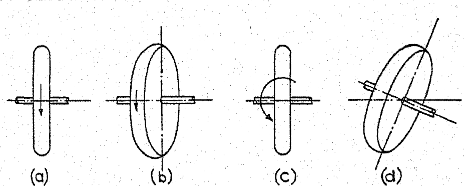


Fig. 3

Fig. 3 (*a*) shows the front view of a rotating wheel. This has angular velocity about the axle, but none about an axis perpendicular to the plane of the paper. If the plane of the wheel is turned so that

it appears as in Fig. 3 (*b*), then the angular velocity has a component about the axis perpendicular to the plane of the paper.

This is also shown in plan view in Fig. 4, *AA* being the axis in question. In order to produce this angular velocity, a moment must be supplied about the axis *AA* as shown in Fig. 3 (*c*). Thus a moment about one axis produces rotation about another, when the gyroscope turns about a third. If the moment shown in Fig. 3 (*c*) is not available, the wheel will rotate as shown in Fig. 3 (*d*) fast enough to keep the total angular momentum zero about the axis perpendicular to the plane of the paper.

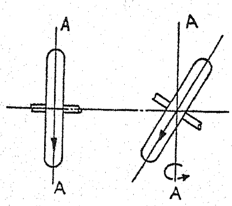


FIG. 4

If a gyroscope has a moment of inertia I and angular velocity ω, its angular momentum is $I\omega$. If it turns about an axis perpendicular to the shaft, with an angular velocity Ω, the torque required to produce this motion is

$$\text{Torque} = \frac{I}{g}\,\omega\Omega$$

about a third axis perpendicular to the other two. For example, a wheel 36 in. in diameter weighs 300 lb., has a radius of gyration of 15 in. and is running at 90 m.p.h.

$$I = 300 \times \left(\tfrac{15}{12}\right)^2 = 469 \text{ lb.-ft.}^2$$

$$\omega = \frac{132 \text{ ft./sec.}}{1.5 \text{ ft.}} = 88.0 \text{ radians/sec.}$$

If this wheel changes its direction at a rate of 0.15 radians/sec., the moment required to hold the wheel up is

$$\frac{468}{32.2} \times 88.0 \times 0.15 = 192 \text{ lb.-ft.}$$

Chapter 2

VIBRATIONS OF SYSTEMS WITH ONE DEGREE OF FREEDOM

A great many vibration problems come within the simple theory which will be given in this chapter and which is the basis for most of the later work.

In Fig. 5 a weight is supported by a light spring. The weight can vibrate up and down on the spring. If we know the height of the

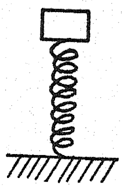


Fig. 5

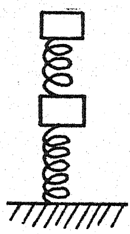


Fig. 6

weight at any moment we know the position of the whole system and since this one distance fixes everything, the system is said to have one degree of freedom. If we had two weights as in Fig. 6 it would be necessary to know the heights of both weights before we could tell the

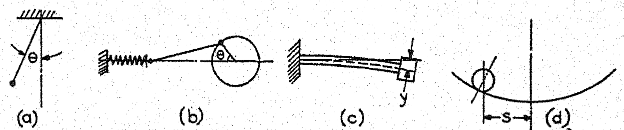


Fig. 7

position of everything. For this reason this system is said to have two degrees of freedom. Other systems with one degree of freedom are shown in Fig. 7, in which (*a*) shows a pendulum whose position is

determined by the angle θ, (*b*) shows a crank and connecting rod operating against a spring. The position of the whole system can be found if the crank angle θ is known. (*c*) shows a weight on a light cantilever, the position being fixed by the sag of the weight; (*d*) shows a ball rolling in a track, the distance from the center determining the position completely.

In the great majority of practical cases of vibration there is one mass which can vibrate about its normal position of equilibrium and the force which acts on the mass is approximately proportional to the displacement from this equilibrium position.

For example, the weight on the spring, the pendulum and the weight on the cantilever are all covered by this description. If a system of this kind is set vibrating and then left to itself, the mass will vibrate according to the equation

$$y = A \cos 2\pi ft$$

where y is the distance from the equilibrium position, A is the greatest distance, f is the frequency of vibration (complete double swings per sec.), t is the time, measured from some instant at which the mass is at the extreme position of its swing. If T is the period of the vibration, that is, the number of seconds required for a complete swing to and fro,

$$T = \frac{1}{f}.$$

It is very convenient to use one letter, *omega*, to represent

$$\omega = 2\pi f = \frac{2\pi}{T}$$

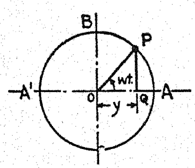


FIG. 8

and ω is called the angular velocity of the motion for the following reason. In Fig. 8 $A'OA$ is a diameter of a circle.

A point P starts at A and moves anticlockwise around the circle with constant angular velocity ω. PQ is a perpendicular from P onto $A'OA$. Then $OQ = OP \cos \omega t$ so that if OA is of length A, and OQ is of length y,

$$y = A \cos \omega t.$$

This representation is of great use in dealing with motions of this kind. The motion of Q is called simple harmonic, because a pure tone in music is produced by vibrations of the same sort. In a simple harmonic motion,

$$y = A \cos \omega t.$$

$$\begin{array}{lll} \text{Maximum displacement} & = & A \\ \quad\text{''} \quad \text{velocity} & = & A\omega \\ \quad\text{''} \quad \text{acceleration} & = & A\omega^2. \end{array}$$

The value of the angular velocity ω is obtained from the formula

$$\omega = \sqrt{\frac{\text{restoring force per unit displacement}}{\text{weight of the vibrating part}} \times g}$$

where g = acceleration due to gravity = 32.2 ft./sec./sec.

Example. A weight W supported on a light spring is in equilibrium with the spring compressed a distance of "a" feet under the weight of W lb. Hence the spring exerts a force W lb. for a displacement a and

$$\frac{W}{a} \text{ lb. for a unit displacement.}$$

Therefore the angular velocity corresponding to a vibration is

$$\omega = \sqrt{\frac{\left(\frac{W}{a}\right)}{W} \times g} = \sqrt{\frac{g}{a}}$$

from which the frequency is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{a}}$$

and the period is

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{a}{g}}$$

The period of vibration of a weight on a spring therefore depends only on the normal or static deflection of the spring.

A car mounted on springs which deflect 4 in. under the weight will vibrate vertically with a period

$$T = 2\pi\sqrt{\frac{\left(\frac{4}{12}\right)}{32.2}} = 0.64 \text{ sec.}$$

In all practical applications, a system like this, set vibrating and left to itself, will not continue to vibrate indefinitely, but the vibration will subside more or less quickly, due to friction, damping or other losses.

Friction is generally supposed to be represented by an approximately constant force-resisting motion. Damping due to small movements in air or fluids or between lubricated parts is taken as producing a retarding force proportional to the velocity. This happens to be a convenient kind of force to deal with mathematically and therefore "velocity damping" is assumed wherever the assumption is at all reasonable.

Internal friction in materials is often important. It varies with the material and with the maximum stress during the vibration. The vibrations referred to above are "free." That is, the system is left to itself after the vibration is started. In many cases the vibrations are produced by some regular disturbing force and are called "forced." For example, take the weight supported by a spring, which has been considered above. If the spring is on a wheel which rolls over a washboard road as in Fig. 9 (*a*), the weight will vibrate up and down with the road surface. The faster the system travels, the faster will be the vibration. As the frequency of vibration approaches the natural frequency of free vibration $f = \frac{1}{2\pi}\sqrt{\frac{g}{a}}$ the amount of vibration becomes greater, as in Fig. 9 (*b*). When the frequencies coincide, the vibration is a maximum and is limited only by whatever friction may be in the spring, if the spring does not go solid or break. This is a resonant vibration, shown in Fig. 9 (*c*). It will be seen that the weight reaches its highest points *after* the peaks in the road surface, instead of at them, and the vibration is said to lag behind the road wave.

At still higher speeds the vibration decreases until at very high

speed the weight will travel smoothly in an almost straight line and all the road irregularity will be taken up in the spring. Motion above the resonant speed is shown in Fig. 9 (*d*). The vibration lags half a

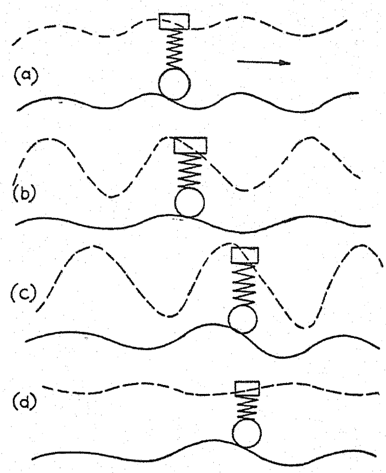


FIG. 9

wave, so that the weight is highest when the road surface is lowest and vice versa.

The relation between the amount of a forced vibration and its frequency is shown in Fig. 10. The relation is

$$e = \frac{1}{1 - \left(\frac{f}{f_c}\right)^2}$$

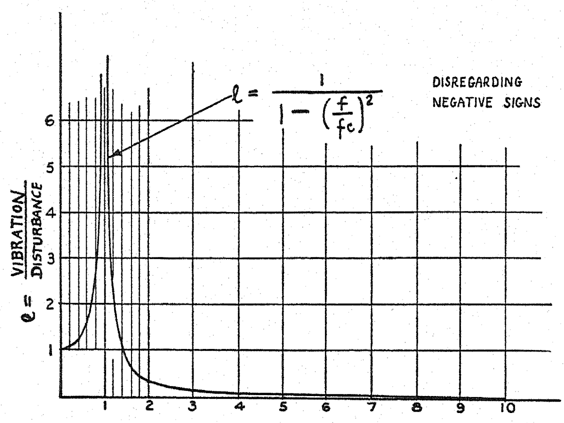
where $e = \dfrac{\text{Amplitude of forced vibration}}{\text{Amplitude of the disturbance}}$

f = Frequency of the disturbance (and of the forced vibration)

f_c = Natural "free" frequency of the system.

When the forced frequency coincides with the natural frequency, $f = f_c$, the formula indicates that the forced vibration becomes

infinite. This is of course prevented by damping or mechanical interferences. When the forced frequency f is greater than the natural frequency f_c, the formula shows e to be negative, which simply means that the vibrations are in opposite directions as in Fig. 9 (*d*).



$$\frac{f}{f_c} = \frac{\text{Frequency of disturbance}}{\text{Natural frequency}}$$

FIG. 10

Example. A short car truck has springs which deflect 6 in. under the weight of the car. The truck runs on track with low rail joints every $19\frac{1}{2}$ ft. Assuming that the track surface may be taken as a sine curve with a total variation of level of $\frac{1}{4}$ in., find the critical speed for resonant vertical vibration and also the amount of vibration at 70 m.p.h.

The natural frequency is

$$f_c = \frac{1}{2\pi}\sqrt{\frac{g}{a}}$$

where $g = 32.2$ ft./sec./sec.

$$a = 6 \text{ in.} = \tfrac{1}{2} \text{ ft.}$$

$$\therefore f_c = \frac{1}{6.28}\sqrt{\frac{32.2}{(\frac{1}{2})}} = 1.28 \text{ vibrations per sec.}$$

There will be resonance when the frequency with which the car passes rail joints is equal to the frequency of its natural vibration. That is, when the car travels

$$19.5 \times 1.28 \text{ ft. per sec.}$$

$$= 24.9 \text{ ft. per sec.}$$

$$= 17 \quad \text{m.p.h.}$$

The frequency of rail joints at 70 m.p.h. will be 70/17 times the frequency at 17 m.p.h. Therefore, $\frac{f}{f_c} = 4.12.$

From Fig. 10 the vibration is 0.063 times the disturbance, so that at 70 m.p.h. the car will vibrate with a total amplitude of

$$\tfrac{1}{4} \times .063 = 0.016 \text{ in.}$$

and the frequency will be $\dfrac{5280 \times 70}{3600 \times 19.5} = 5.27$ vibrations per sec.

The ratio e of $\dfrac{\text{vibration}}{\text{disturbance}}$ is called the "transmissibility" of the spring support or sometimes the "magnification factor" or "dynamic magnifier." It applies to the force transmitted through a system as well as to the motion.

Example. A motor runs at 1800 r.p.m. and has a maximum unbalanced force of 25 lb. It is required to design a spring support which will reduce the vibration force on the base to a maximum of 1 lb.

The transmissibility of the support must be 1/25, whence, from Fig. 10 or from

$$\frac{1}{25} = -\frac{1}{1 - \left(\dfrac{f}{f_c}\right)^2}$$

$$\frac{f}{f_c} = \sqrt{25 + 1} = 5.1.$$

The forced frequency is 1800 per min., or 30 vibrations per sec., so that the natural frequency must be

$$f_c = \frac{30}{5.1} = 5.9 \text{ vibrations per sec.}$$

But, if the spring support has a static deflection "a" feet

$$f_c = \frac{1}{2\pi}\sqrt{\frac{g}{a}} = 5.9$$

from which

$$a = \frac{g}{(2\pi f_c)^2} = \frac{32.2}{(6.28 \times 5.9)^2} = 0.024 \text{ ft.} = 0.28 \text{ in.}$$

Note particularly that too stiff a spring mounting will increase the transmission of vibration and that it may be better to use a rigid mounting rather than one which is too stiff. As will be seen from Fig. 10, the spring reduces the vibration only if $\frac{f}{f_c}$ is greater than 1.41; that is, if the natural frequency is less than 70% of the forced frequency.

Damping of Forced Oscillations. The addition of damping to a spring support has an effect which varies according to the ratio of the forced frequency to the natural frequency. This is readily seen from the fact that *a very heavily damped spring is practically a rigid support.* For forced frequ ncies near resonance, damping is generally essential in order to keep the forced vibrations from becoming too large. For forced frequencies greater than 1.41 times the natural frequency, damping *increases* the transmissibility of vibration and is detrimental unless it is required for some other reason.

Therefore, for cases of steady forced vibration, when a spring support reduces the transmissibility, damping increases it and vice versa. It follows that springs should only be damped to take care of some condition other than the steady vibration. For example, in starting or stopping, a machine may pass through speeds at which its spring support becomes resonant and damping may be required to prevent undue vibration during starts and stops.

There is always some damping in practice, but it is often so small that it can be neglected in steady vibrations except at the point of resonance.

Damping generally has very little effect on the natural frequency of a system unless the damping is so great that vibrations are no longer possible and if the system is disturbed it merely subsides back to its equilibrium position.

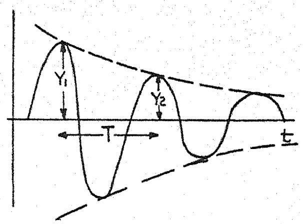


FIG. 11

Let $b = \dfrac{\text{Damping force per unit of velocity}}{\text{Weight which is being damped}} \times g$

ω_0 = angular velocity corresponding to the vibration without damping

ω = angular velocity corresponding to the vibration with damping

Then $$\omega = \sqrt{\omega_0{}^2 - \left(\frac{b}{2}\right)^2} = 2\pi f = \frac{2\pi}{T}.$$

The ratio of one swing to the succeeding one,

$$d = \frac{y_1}{y_2} \text{ in Fig. 11;} \quad \text{is } e^{\frac{b}{2}T}$$

where $e = 2.71828\ldots$ the base of natural logarithms.

It is common to use the ratio

$$\delta = \frac{b}{2}T = \frac{b}{2f}$$

to measure the damping and since this is the natural logarithm of the ratio of successive swings it is called the logarithmic decrement or simply the decrement.

In Fig. 12 curve 1 shows the relation of frequency to decrement and curve 2 shows the relation of the ratio of successive swings to decrement. It will be clear that with a decrement of 0.7 for which each swing is half the preceding one, the frequency is only decreased

0.6% by the damping. Even for a decrement of 2.3 for which each swing is $\frac{1}{10}$ the preceding one, the frequency is only decreased 4.1%.

Starting of a Forced Oscillation. The steady forced oscillation already studied did not contain anything to show how the oscillation

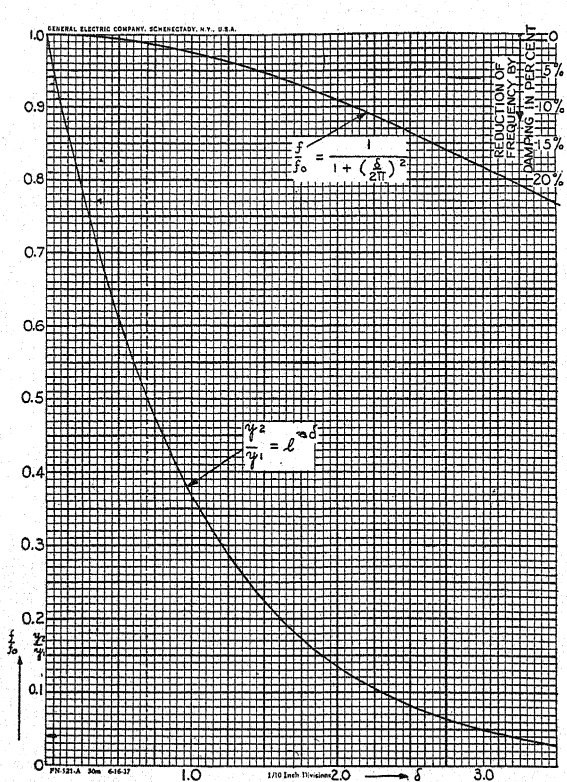
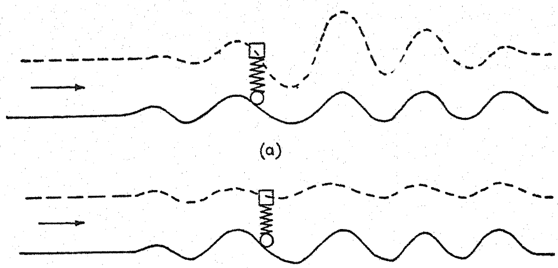


FIG. 12. DAMPING MEASURED BY LOGARITHMIC DECREMENT δ

had been started. It was assumed that any irregularities due to starting had been damped out. When a forced harmonic motion starts, it can be considered as made up of two simultaneous motions. The first is the steady forced motion which has been discussed. The second is a free vibration with the natural frequency of the system.

If there is any damping at all in the system, it will gradually damp out the free vibration and leave only the forced.

If the system is at rest in the equilibrium position when the disturbing force first acts, the free and forced vibrations must start by being equal and opposite. The ensuing motion will appear irregular because the free and forced vibrations have different frequencies. If the frequencies are close together there will be "beats." That is, the free and forced vibrations will first subtract, then, after a while, they will add and the result will be similar to that shown in Fig. 13 (*a*).



(a) (b)

FIG. 13

If the motion is slightly damped, the free vibration will die out, as in Fig. 13 (*b*).

Disturbing Forces Which Are Not Sine Waves. Thus far it has been assumed that the disturbance can be represented by some curve like

$$y = A \cos \omega t.$$

If this is not the case, it can always be represented by a series of such expressions. For example, the motion of a piston driving a crank at constant speed is not exactly a sine curve (or cosine curve: the only difference is the moment from which the time is measured).

The distance of the piston from the center of rotation of the crank, O in Fig. 14, is approximately

$$x = l\left(1 - \frac{r^2}{4l^2}\right) + r\left(\cos \omega t + \frac{r}{4l}\cos 2\omega t\right).$$

Thus the position and therefore also the inertia force due to the piston are made up of a cos ωt term and a cos $2\omega t$ term. If the engine is

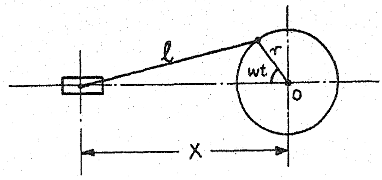


FIG. 14

producing vibrations in its support there will be two series of vibrations, one with the fundamental frequency $f = \dfrac{\omega}{2\pi}$ and another of twice this frequency.

Any practical curve of force or movement can be split up into a fundamental sine curve and a series of others of 2, 3, 4, etc., times its frequency, which are called the harmonics.

Spring Constants Which Vary with Time. Consider a spring-supported weight, rolling on a rail with regular joints. Suppose the joint is more flexible than the solid rail, so that if the weight rolls very slowly it will drop and rise as it goes over each joint, as shown in Fig. 15 (*a*), (*b*) and (*c*). The dotted line in Fig. 15 (*d*) shows the

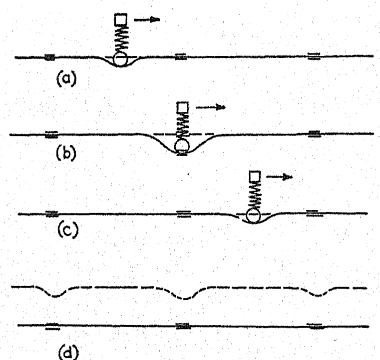


FIG. 15

path of the weight for very slow motion. If the path of the wheel were the same at speed as it is in Fig. 15 (*d*) we would have a simple

case of forced vibration which has been studied above and the fundamental frequency would be that with which the wheel passed over the joints. However, the path of the wheel is not fixed; it depends on the pressure of the wheel on the rail. Below the resonant speed, the wheel will follow the slow speed path and will drop into all the joints. As the speed approaches the resonant speed, the vibration will lag and the wheel will begin to follow a path corresponding to the dotted line in Fig. 16 (*a*).

A slight increase of speed will produce more lag, until the wheel hardly drops into the joint at all, as in Fig. 16 (*b*). It will be seen

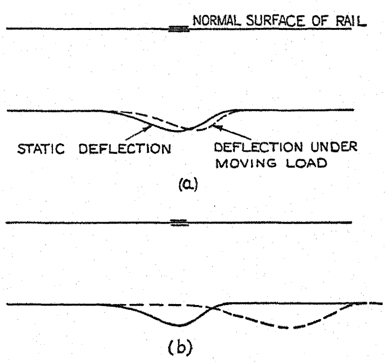


FIG. 16

that in Fig. 16 (*a*) the greatest deflection is at the flexible part of the rail while in Fig. 16 (*b*) the greatest deflection is at the stiff part of the rail. The average stiffness which resists the motion is therefore greater in Fig. 16 (*b*) than in Fig. 16 (*a*) and the frequency will be higher. In other words, there is no longer one critical speed. There is a small range of speed with resonance over the whole range. As the speed increases through this range the vibration lags behind the joints from zero to a greater period. The same applies to forced vibrations of 2, 3 or more times the fundamental frequency. In each case there is a resonant range, over the whole of which the vibration increases indefinitely until limited by damping, etc.

There is a further difference resulting from this variable stiffness. In Fig. 17 a vibration is shown which corresponds to every other joint.

It will be evident that the joints B and D are effective in sustaining the vibration. At A, C and E, however, there is very little deflection of the rail and as a result the flexibility of these points is of little or no importance. In fact, the rail behaves just as if it had joints only at B and D. To express the idea simply, if the wheel jumps over every other joint, it has no way to tell that those joints are there. The result is that resonance is possible when the natural frequency coincides with the frequency with which the wheel passes *two* joints.

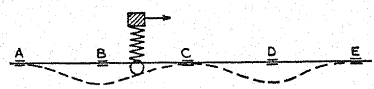


FIG. 17

This is at a speed twice the fundamental and is therefore likely to be of importance.

The side-rods of electric locomotives are also systems whose stiffness varies with the position of the cranks. They are subject to vibrations of the same character as those described above.

Non-linear Systems. In all the cases so far considered the restoring forces have been proportional to the displacement, even if the stiffness varied with time, and the damping forces have been proportional to the velocity. This large and very important class of vibrating systems is called linear. If the restoring force is not proportional to the displacement or if the damping is not proportional to the velocity, then the system is said to be non-linear. Systems with clearances between the parts or with constant friction are non-linear. The most important distinction which should always be remembered is that vibrations of a linear system are independent of each other and can be superposed one on another. Vibrations of a non-linear system are not independent. They affect each other. For example, a single cylinder engine produces forces varying as $\cos \omega t$ and as $\cos 2\omega t$ as noted above. If this engine is supported on a linear system, it is possible to calculate the effects of the $\cos \omega t$ and $\cos 2\omega t$ terms separately and to add the results. If the support is non-linear—for example, if it is damped by dry rubbing surfaces, with constant friction—the $\cos \omega t$ and $\cos 2\omega t$ vibrations are not separate, but each affects the other. Naturally it is generally much easier to deal with

linear systems because every harmonic force or motion can be studied or calculated separately. Therefore ways have been devised to imitate non-linear systems with so-called equivalent linear systems in order to make calculations easy. This is a good method so long as it is always remembered that the non-linear system is fundamentally different and that all results obtained from "linear" calculations must be carefully examined to see whether or not they can be accepted as reasonably accurate.

Self-sustaining Oscillations. Resonant oscillations fall in this class and have already been considered, but there are numerous others where an oscillation builds up and persists in the absence of any regular external disturbing force.

The nosing of car trucks and of locomotives and the front-wheel shimmy of automobiles are good examples of such oscillations.

In other fields the hunting of governors or of electrical machines and the vibrations of wires in a wind are common examples.

Every self-sustained oscillation must have a source from which it draws energy to make up for the energy which is dissipated in friction or damping. A nosing car truck or a shimmying automobile draws its energy from the force which is required to keep it moving. A wire vibrating in the wind draws its energy from the wind. Most problems of this sort which concern vehicles are rather complex, involving several degrees of freedom and they will therefore be postponed to later sections.

Chapter 3

SIMPLE HARMONIC MOTION (*Continued*)

Vector Representation. In Fig. 18, P starts at A and moves counterclockwise round the circle ABA' with constant angular velocity ω. PQ is a perpendicular from P onto $A'OA$. Then $OQ = OP \cos \omega t$ so that if OA is of length A and OQ is of length y.

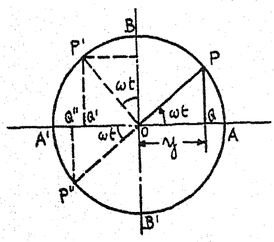


FIG. 18

$$y = A \cos \omega t$$

The maximum velocity of Q in the positive direction occurs when P is at B' and Q passes through O and is then equal to the velocity of P, which is $A\omega$. The maximum positive acceleration of Q towards A occurs when P and Q are at A' and is then equal to the acceleration of P, which is $A\omega^2$. It will be seen that if OP' is drawn at right angles to OP, 90° ahead of OP, and if OP'' is drawn 180° ahead of OP and perpendiculars $P'Q'$ and $P''Q''$ are drawn onto $A'A$,

the displacement of Q is $OQ \qquad = A \cos \omega t$

the velocity of Q is $\omega \times OQ' \quad = -A\omega \sin \omega t$

the acceleration of Q is $\omega^2 \times OQ'' = -A\omega^2 \cos \omega t.$

It is convenient to represent the entire motion by the line OP, its length representing the maximum displacement and its angle with OA representing the phase of the motion, that is, the place in the cosine wave at which we start to measure the time. Thus a line OP (Fig. 19) inclined at an angle β to the base OA represents a motion $OP \cos (\omega t + \beta)$.

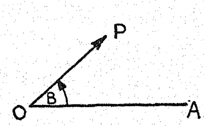


FIG. 19

It is understood that OP is rotating with constant angular velocity

ω around O, that B determines its position when the time is zero and that the actual motion is represented by the projection of OP on the base OA.

A line such as OP which has magnitude and direction is called a vector. The velocity of the motion will then be represented by a

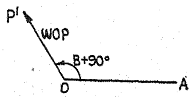


FIG. 20

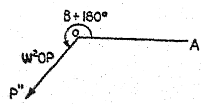


FIG. 21

vector OP' of length ωOP rotated 90° ahead of OP as in Fig. 20. Likewise the acceleration is represented in Fig. 21 by a vector OP'' of length $\omega^2 OP$, rotated 180° ahead of OP. The actual velocity and acceleration are obtained by taking the projections of OP' and OP'' on OA.

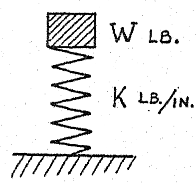


FIG. 22

Proof That a Weight on a Spring Vibrates in Simple Harmonic Motion. So far this has simply been stated as a fact.

Referring to Fig. 22, a weight W lb. rests on a spring whose stiffness is k lb. per in. deflection.

If the weight is deflected a distance y from the position of equilibrium, the force tending to return the weight to the equilibrium position is ky lb. (tending to decrease y).

$$\text{But force} = \text{mass} \times \text{acceleration}.$$

$$\text{Therefore } -ky = \frac{W}{g} \times \text{acceleration}.$$

$$\therefore \text{Acceleration} = -\frac{kg}{W}y.$$

But it was shown above that if OP represents the motion y, then the acceleration is OP'' equal to $\omega^2 OP$ and directly opposed to OP.

$$\text{That is, acceleration} = -\omega^2 y$$

Hence the equation of motion is satisfied if the weight vibrates in simple harmonic motion if

$$\omega^2 = \frac{kg}{W}$$

or

$$\omega = \sqrt{\frac{kg}{W}}$$

which is the formula given in Chapter 2.

Given the Starting Conditions, Find the Resulting Motion. If a vibration is simple harmonic, OP represents the motion and OP' represents the velocity and $OP' = \omega OP$ (see Fig. 23).

The actual values are the projections of these vectors on the base AA' so that

$$\text{displacement} = OQ$$

$$\text{velocity} = OQ'.$$

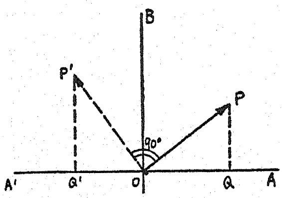


FIG. 23

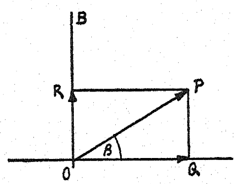


FIG. 24

Now project OP onto OB and let the projection be OR (see Fig. 24). The triangle OPR can be obtained from the triangle $OP'Q''$ by rotating it 90° round O until OP' lies along OP and then reducing each side to $\frac{1}{\omega}$ of its original size, so that P' will lie on P. This shows that if OQ' represents the velocity, OR will represent $-\frac{1}{\omega} \times$ the velocity. This gives a method of finding the motion if the original position and velocity are given, because OQ and OR are known and OP can be drawn.

Example. A weight of 2000 lb. rests on a spring which deflects 3 in. under the load. At a particular moment the weight is 1 in. above

the static position and is going down at a rate of 2 in. per sec. Find the motion.

First we have the angular velocity

$$\omega = \sqrt{\frac{g}{\text{static deflection}}}$$

$$= \sqrt{\frac{32.2}{\left(\frac{3}{12}\right)}} = 11.4 \text{ radians/sec.}$$

$$\text{The frequency} = \frac{\omega}{2\pi} = \frac{11.4}{6.28} = 1.81 \text{ vibrations per sec.}$$

The displacement is $+1$ in.
The velocity is -2 in. per sec.

$$-\frac{1}{\omega} \times \text{the velocity is } -\frac{1}{11.4} \times (-2) = 0.176.$$

Hence $\quad OQ = 1$ in.

$OR = 0.176$ in.

and $\quad OP = \sqrt{OQ^2 + OR^2} = 1.015$ in.

It is evident that $\tan \beta = \dfrac{QP}{OQ} = \dfrac{OR}{OQ} = 0.176.$

Therefore $\beta = 10° = 0.175$ radians.

If y represents the displacement in inches above the static position,

$$y = OP \cos(\omega t + \beta)$$
$$= 1.015 \cos(11.4\,t + .175) \text{ in.}$$

If OR or OQ is negative, it is drawn in the opposite direction.

Forced Oscillations. In Fig. 25 OP represents a forced harmonic motion, as of a weight on a spring, with a force $F \cos \omega t$ acting on the weight as in Fig. 26.

The acceleration is $-\omega^2 OP$ and

$$\text{Mass} \times \text{acceleration} = -\frac{W}{g}\,\omega^2 OP$$

which is represented by OT.

The force acting on the weight to increase the displacement OP is $-k \times OP$ due to the spring and F due to the external force. Hence, since mass $\times$ acceleration = force,

$$-\frac{W}{g}\omega^2 OP = -k \times OP + F.$$

Therefore
$$OP = \frac{F}{k - \frac{W}{g}\omega^2}.$$

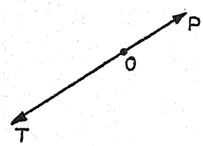


FIG. 25

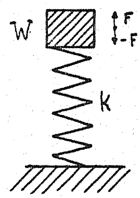


FIG. 26

But if ω_c = the natural angular velocity corresponding to free vibration,

$$\omega_c^2 = \frac{kg}{W}$$

$$\therefore\ OP = \frac{F}{k} \times \frac{1}{1 - \left(\frac{\omega}{\omega_c}\right)^2}.$$

The vector method is particularly useful in dealing with forced oscillations of a damped system.

If the weight is damped with a force which is equal to λ times the velocity,

Mass $\times$ acceleration = force

= spring force + damping force + external force

= k displacement $- \lambda \times$ velocity + force

or, ($k \times$ displacement) + ($\lambda \times$ velocity)+

$$\left(\frac{W}{g} \times \text{acceleration}\right) = \text{force.} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But if the vector OP in Fig. 27 represents the displacement, $OP' = \omega OP$, 90° ahead of OP, represents the velocity, and $OP'' = \omega^2 OP$, 180° ahead of OP, represents the acceleration. Hence, starting at O in Fig. 28 we draw $OA = kOP$.

$$\text{Add the vector } AB = \lambda OP' = \lambda\omega OP$$

$$\text{Add the vector } BC = \frac{W}{g} OP'' = \frac{W}{g} \omega^2 OP.$$

Then the resultant, OC, must be the left-hand side of equation (1) and must equal the force, that is, $OC = F$.

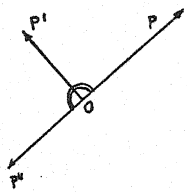


FIG. 27

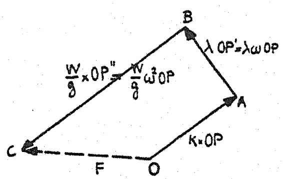


FIG. 28

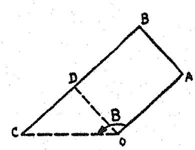


FIG. 29

Redrawing the diagram in Fig. 29 and drawing OD perpendicular to BC, it will be seen that ODC is a right-angled triangle and hence

$$OC^2 = OD^2 + DC^2.$$

But
$$OC = F$$
$$OD = AB = \lambda\omega OP$$
$$DC = BC - BD$$
$$= BC - OA$$
$$= \frac{W}{g}\omega^2 OP - kOP = \left(\frac{W}{g}\omega^2 - k\right) OP.$$

Hence
$$F^2 = (\lambda\omega OP)^2 + \left(\frac{W}{g}\omega^2 - k\right)^2 OP^2$$
$$= (\lambda\omega)^2 OP^2 + k^2\left(1 - \frac{\omega^2}{\omega_c^2}\right)^2 OP^2$$

or
$$OP = \frac{F}{k} \times \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_c^2}\right)^2 + \left(\frac{\lambda\omega}{k}\right)^2}}$$

If there is no damping, λ becomes zero and this reduces to

$$\frac{F}{k} \times \frac{1}{1 - \left(\frac{\omega}{\omega_c}\right)^2}$$

as before.

In Fig. 28 the vectors represent all the forces and considerable information can be obtained by inspection of the diagram. Thus the damping force is represented by the vector $AB = \lambda\omega OP$.

$$= \frac{F}{k} \times \frac{\lambda\omega}{\sqrt{\left(1 - \frac{\omega^2}{\omega_c^2}\right)^2 + \left(\frac{\lambda\omega}{k}\right)^2}}$$

At the point of resonance, where

$$\omega^2 = \frac{gk}{W}$$

$$BC = \frac{W}{g}\,\omega^2 OP = kOP = OA$$

and the vector diagram becomes a rectangle, and

$$OP = \frac{F}{k} \times \frac{1}{\left(\frac{\lambda\omega}{k}\right)} = \frac{F}{\lambda\omega}$$

Thus, when the disturbing force has the same frequency as the natural vibration, the amount of the vibration is limited only by damping and the vibration lags 90° behind the disturbing force.

The angle which the vibration lags behind the disturbing force is

$$\beta = COA$$

$$\tan\beta = -\frac{AB}{DC} = -\frac{\lambda\omega OP}{\left(\frac{W}{g}\,\omega^2 - k\right) OP} = \frac{\frac{\lambda\omega}{k}}{1 - \frac{\omega^2}{\omega_c^2}}$$

It will be seen that the vibration of the weight lags slightly behind the disturbing force when the frequency is low. As the frequency increases, the lag increases until at resonance it is 90°. At frequencies above resonance the lag increases until it approaches 180°.

Consider a weight supported on a damped spring which travels over a wavy road as in Fig. 30. Let the height of each crest and the depth of each hollow be "a" from the average level of the road.

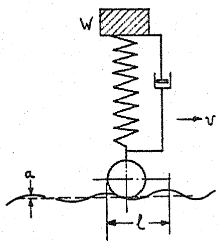


FIG. 30

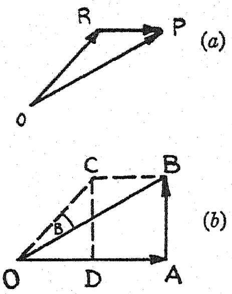


FIG. 31

In Fig. 31 (a) suppose that $OR = a$ is the vibration of the wheel up and down. Suppose that RP is the extension of the spring, then their sum, OP, is the vibration of the weight.

As above, in equation (1),

$$\begin{aligned}&(k \times \text{spring defl.})\\&+ (\lambda \times \text{relative velocity})\\&+ \left(\frac{W}{g} \times \text{acceleration}\right)\\&= \text{External force acting on the weight} = 0 \quad . \quad . \quad . \quad (2)\end{aligned}$$

In Fig. 31 (b) draw

$$OA = k \times RP$$

$$AB = \lambda\omega RP, 90^\circ \text{ ahead of } RP$$

$$BO = \frac{W}{g}\,\omega^2 OP, \; 180^\circ \text{ ahead of } OP.$$

These vectors, representing the three terms on the left of equation (2) must equal zero, that is, they must form a triangle, as shown.

Then draw $$OC = \frac{W}{g}\,\omega^2 OR = \frac{W}{g}\,\omega^2 a$$

$$CB = \frac{W}{g}\,\omega^2 RP$$

and since $$OB = \frac{W}{g}\,\omega^2 OP,$$

the triangles ORP and OCB are similar.

Draw CD perpendicular to OA

Then $$OC^2 = OD^2 + DC^2$$

$$= (OA - CB)^2 + AB^2$$

Therefore $$\left(\frac{W}{g}\,\omega^2 a\right)^2 = \left(kRP - \frac{W}{g}\,\omega^2 RP\right)^2 + (\lambda\omega RP)^2$$

$$\therefore\ RP = \frac{W\omega^2 a}{gk} \times \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_c^2}\right)^2 + \left(\frac{\lambda\omega}{k}\right)^2}}$$

$$= \frac{a\omega^2}{\omega_c^2} \times \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_c^2}\right)^2 + \left(\frac{\lambda\omega}{k}\right)^2}}$$

This is the vibration of the spring.

The vibration of the weight is represented by OP, which is obtained from the equation

$$OB^2 = OA^2 + AB^2$$

From which

$$\left(\frac{W}{g}\,\omega^2 OP\right)^2 = k^2RP^2 + \lambda^2\omega^2RP^2$$

$$\therefore\ OP = \frac{\sqrt{k^2+\lambda^2\omega^2}}{\frac{W}{g}\,\omega^2}\times RP = \frac{\omega_c{}^2}{\omega^2}\sqrt{1+\left(\frac{\lambda\omega}{k}\right)^2}\times RP$$

$$= a \times \frac{\sqrt{1+\left(\frac{\lambda\omega}{k}\right)^2}}{\sqrt{\left(1-\frac{\omega^2}{\omega_c{}^2}\right)^2+\left(\frac{\lambda\omega}{k}\right)^2}}$$

The angle of lag of the weight behind the waves in the road is

$$\beta = COB$$

But $\beta = COA - BOA$

$$\therefore\ \tan\beta = \tan\,(COA - BOA) = \frac{\tan\,COA - \tan\,BOA}{1+\tan COA\,\tan BOA}$$

But $\tan COA = \dfrac{DC}{OD} = \dfrac{AB}{OA - CB} = \dfrac{\lambda\omega RP}{kRP - \frac{W}{g}\,\omega^2RP}$

$$= \frac{\left(\frac{\lambda\omega}{k}\right)}{1-\frac{\omega^2}{\omega_c{}^2}}$$

and $\tan BOA = \dfrac{AB}{OA} = \dfrac{\lambda\omega RP}{kRP} = \left(\dfrac{\lambda\omega}{k}\right)$

$$\therefore\ \tan\beta = \frac{\dfrac{\left(\frac{\lambda\omega}{k}\right)}{1-\frac{\omega^2}{\omega_c{}^2}} - \dfrac{\lambda\omega}{k}}{1+\dfrac{\left(\frac{\lambda\omega}{k}\right)^2}{1-\frac{\omega^2}{\omega_c{}^2}}}$$

$$= \frac{\dfrac{\omega^2}{\omega_c^2}\left(\dfrac{\lambda\omega}{k}\right)}{1 - \dfrac{\omega^2}{\omega_c^2} + \left(\dfrac{\lambda\omega}{k}\right)^2}$$

In the above formulas the terms $\dfrac{\omega}{\omega_c}$ and $\dfrac{\lambda\omega}{k}$ occur frequently. It is generally useful to consider the changes which occur as ω changes since ω corresponds to the speed of a vehicle or of a disturbing machine. Then $\dfrac{\lambda\omega}{k}$ should be written as $\dfrac{\lambda\omega_c}{k} \times \dfrac{\omega}{\omega_c}$ so that changes in ω will affect only the ratio $\dfrac{\omega}{\omega_c}$.

It is also convenient to use the decrement δ, defined above as

$$\delta = \frac{b}{2f_c} = \frac{\lambda g}{W} \times \frac{\pi}{\omega_c} = \frac{\lambda}{k}\frac{gk}{W}\frac{\pi}{\omega_c} = \frac{\lambda}{k}\omega_c^2\frac{\pi}{\omega_c} = \frac{\lambda\omega_c}{k} \times \pi$$

Therefore $\dfrac{\lambda\omega_c}{k} = \dfrac{\delta}{\pi}$.

The characteristics discussed above are illustrated in Figs. 32 (*a*) to 32 (*c*) which will repay careful study. In Fig. 32 (*a*) is shown the forced vibration of a weight on a spring, due to a periodic force. It will be noted that damping decreases the vibration for all values of $\dfrac{\omega}{\omega_c}$ and that for a damping decrement δ which is larger than $\pi\sqrt{2} = 4.44$ no maximum vibration occurs at any value of $\dfrac{\omega}{\omega_c}$ i.e. the vibration decreases with increasing values of $\dfrac{\omega}{\omega_c}$.

In Fig. 32 (*b*) is shown the forced vibration of a similar weight on a spring, due to rolling over a rough road. The curves are similar, but the effect of damping is slightly different. The figure shows that for forced frequencies up to 1.41 × natural frequency, damping reduces the vibration. For higher forced frequencies damping increases the vibration.

In Fig. 32 (*c*) is shown the force in the spring and therefore also the force on the wheel due to vibration.

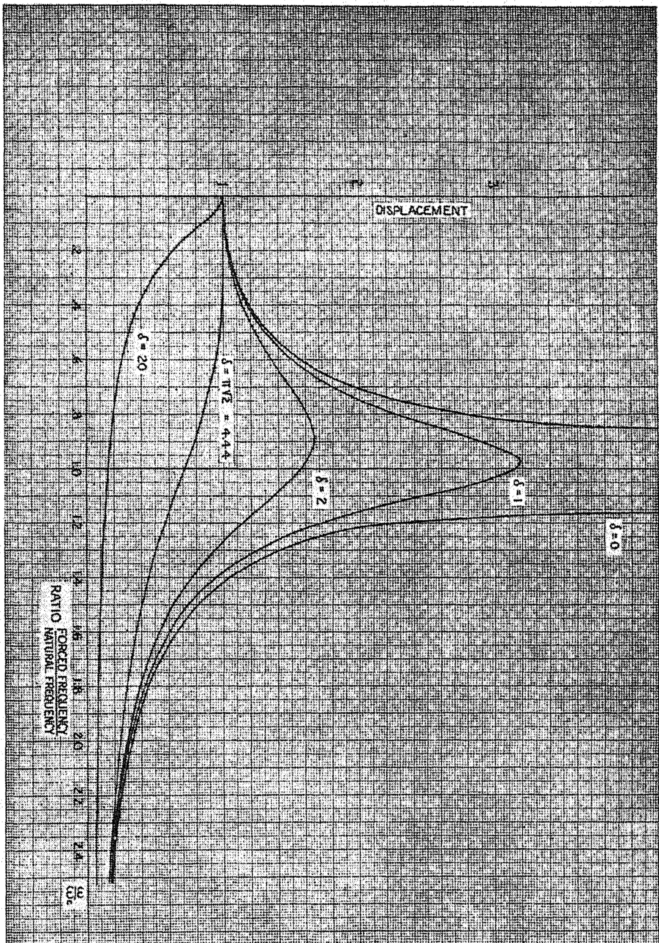
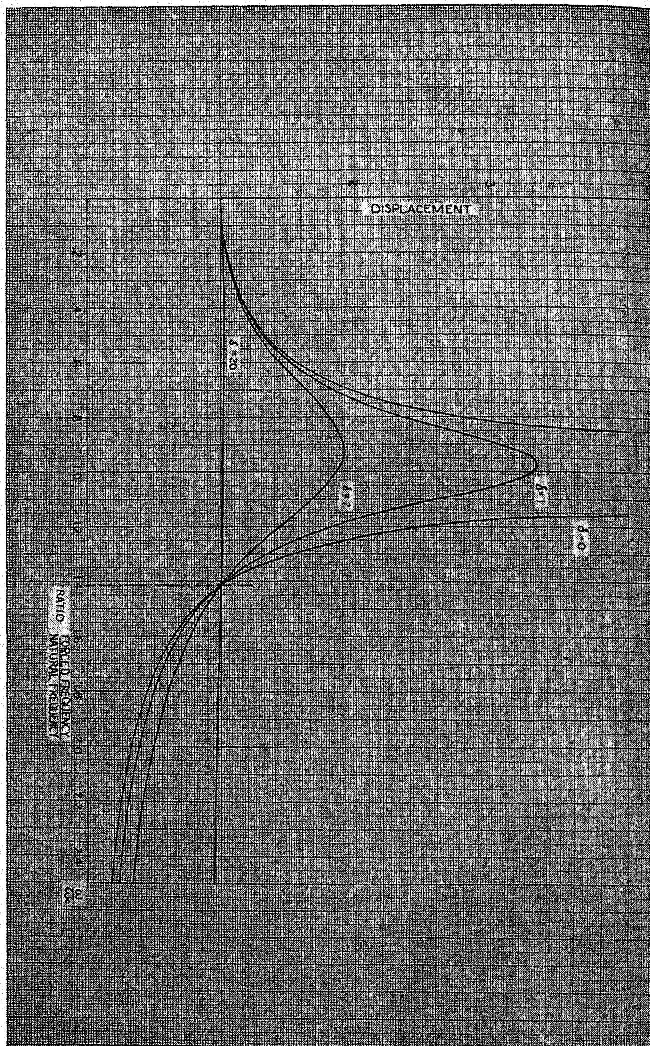
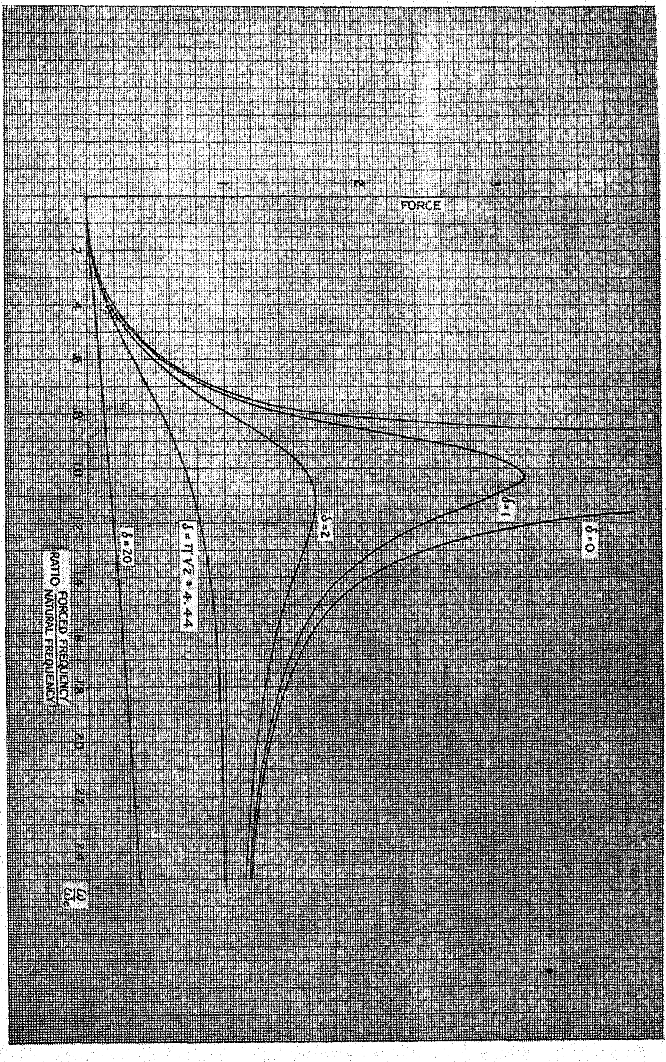

DISPLACEMENT
δ = 20
δ = π√2 = 4.44
δ = 2
δ = 1
δ = 0
RATIO FORCED FREQUENCY / NATURAL FREQUENCY

Fig. 32 (a)



$$\frac{\sqrt{1+\left(\frac{\delta\,\omega_c}{\pi\,\omega}\right)^2}}{\sqrt{\left(1-\frac{\omega^2}{\omega_c^2}\right)^2+\left(\frac{\delta\,\omega}{\pi\,\omega_c}\right)^2}}$$

FIG. 32 (*b*)



$$\frac{\omega/\omega_c^2}{\sqrt{\left(1-\frac{\omega^2}{\omega_c^2}\right)^2+\left(\frac{\delta\,\omega}{\pi\,\omega_c}\right)^2}}$$

FIG. 32 (c)

The total force on the road is made up of the normal load, the force due to vibration of the spring load and the force due to vibration of the unsprung wheel. This corresponds to a number of practical cases.

Effect of a Single Disturbance. In all the cases in the preceding section it has been assumed that the vibration was produced by a regular repeated force and the initial disturbances were neglected, since it was supposed that they had been damped out. It is also useful to study the effect of a single irregularity such as that caused by a spring-supported weight passing over a low spot in the road as in Fig. 33.

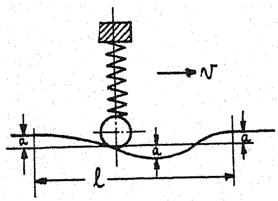


FIG. 33

Let us assume that the low spot has a depth $2a$, the angular velocity of the natural vibration is

$$\omega_c = \sqrt{\frac{g}{\text{static deflection of spring}}}$$

the angular velocity of the forced vibration is $\omega = \dfrac{2\pi v}{l}$

where l = length of the low spot

v = velocity of the weight along the road.

It can be shown that the path of the weight is as shown in Fig. 34 for different values of $\dfrac{\omega}{\omega_c}$. With other things constant, $\dfrac{\omega}{\omega_c}$ is proportional to the speed.

The compression in the spring due to vibration is shown in Fig. 35. It will be noted that the maximum spring compression is given for a value of $\dfrac{\omega}{\omega_c} = 1.5$ and is equal to $2.95\ a$.

In order to obtain the total road pressure, it is necessary to add the normal weight, the extra spring compression due to vibration of the sprung load and the force due to vibration of the unsprung wheel.

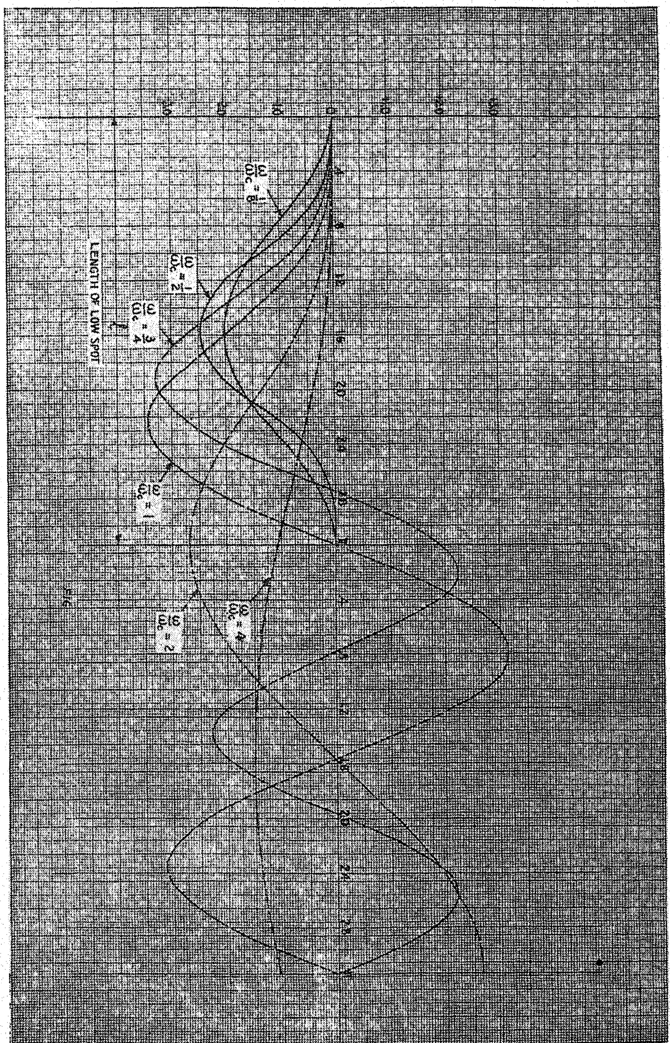


FIG. 34. EFFECT OF A SINGLE DISTURBANCE—PATH OF WEIGHT

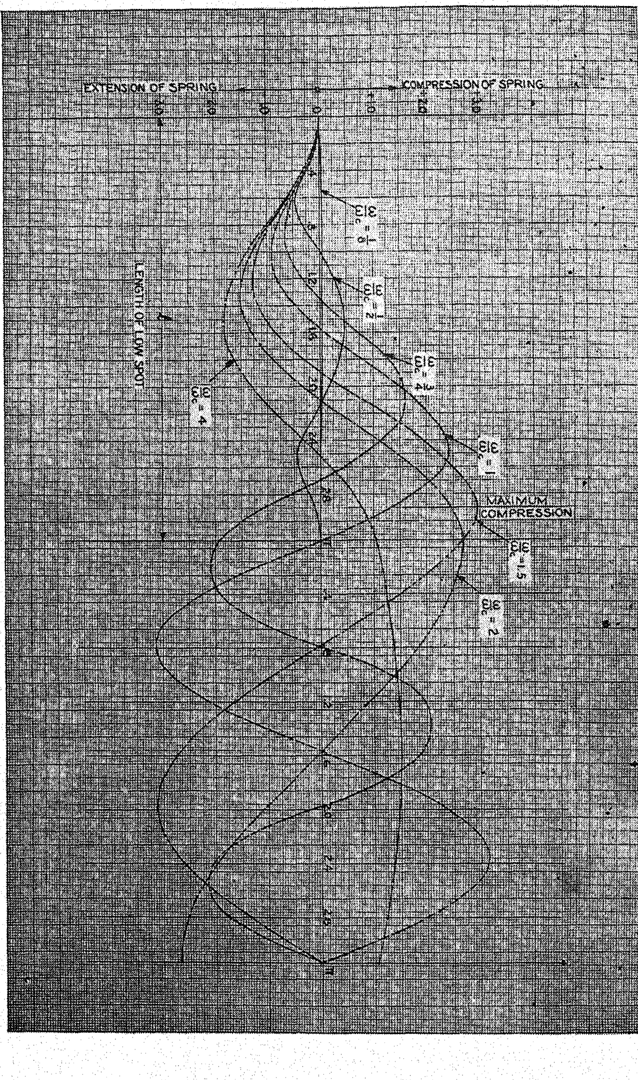


FIG. 35. EFFECT OF A SINGLE DISTURBANCE — COMPRESSION OF SPRING DUE TO VIBRATION

Chapter 4

SYSTEMS WITH TWO DEGREES OF FREEDOM

When it takes two independent measurements to determine the condition of a system at any time, the system is said to have two degrees of freedom. Two systems which are of great importance in the study of vehicle vibrations are shown in Fig. 36. (*a*) shows a weight W, carried on a spring which rests on a second weight W_2 which is also spring-supported. The displacements of both weights must be known in order to determine the condition of the system and these two displacements completely determine it. Fig. 36 (*b*) shows a weight W resting on two springs. The condition will be determined if the displacements of both springs are known. It will also be determined if the position of some definite point in W is known and also the angle which a centerline makes with the horizontal, y and Θ in Fig. 36 (*b*). The condition could be fixed in any number of other ways, but in each case, two distinct measurements are necessary.

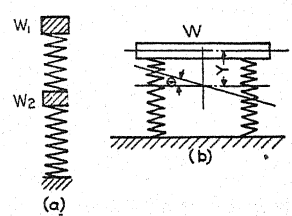


FIG. 36

Systems like this vibrate quite similarly to systems of one degree of freedom, but instead of having one natural frequency they have two distinct frequencies and can be brought into resonance at either. This study will be confined to linear systems; that is, ones in which different vibrations can occur at one time without affecting each other.

Take the system in Fig. 36 (*a*). It can vibrate as in Fig. 37 (*a*) with the two weights moving up and down together with a slow period or it can vibrate as in Fig. 37 (*b*) with the weights always moving in opposite directions and with a faster period.

Both kinds of vibrations can occur at once so that the result may be something like that shown in Fig. 38. The vibrations may appear

quite erratic, but they can be split up into two simple harmonic motions as shown in Fig. 37 (*a*) and (*b*). These are the natural free vibrations of the system. The forced vibrations are similar to those of a simple

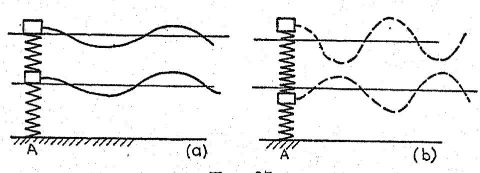


Fig. 37

system. As soon as the free vibrations are damped out, the forced vibration due to a harmonic disturbance is regular.

All points move harmonically and there are two resonance frequen-

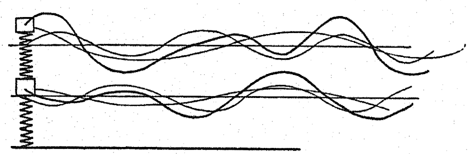


Fig. 38

cies. For instance, in Fig. 39 the system rolls over a wavy road and the ratios of the vibration of the weights to the unevenness of the road are shown in Fig. 40.

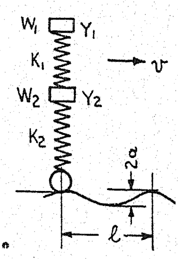


Fig. 39

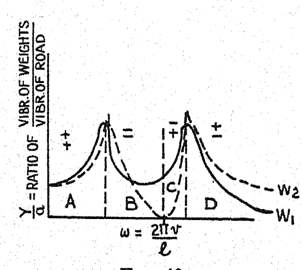


Fig. 40

The full line represents the upper weight W_1 and the chain line represents the middle weight W_2. Over the range A, at slow speed, both weights move up and down together with the waves in the road.

Above the first resonant speed, in the range B, the weights both move opposite to the waves in the road. At still higher speeds, in the range C, the upper weight W_1 moves opposite to the road waves and the lower weight W_2 moves with them. These motions both reverse again above the second resonant speed, in the range D.

If the waves in the road are represented by

$$y = a \cos \omega t = a \cos \frac{2\pi \times vt}{l},$$

the vibration of the upper weight W_1 is

$$y_1 = \frac{a}{\left(1 - \frac{\omega^2}{\omega_1^2}\right)\left(1 - \frac{\omega^2}{\omega_2^2}\right)} \cos \omega t$$

and that of the lower weight, W_2 is

$$y_2 = \frac{a \left(1 - \frac{W_1}{k_1 g} \omega^2\right)}{\left(1 - \frac{\omega^2}{\omega_1^2}\right)\left(1 - \frac{\omega^2}{\omega_2^2}\right)} \cos \omega t$$

Here ω_1^2 and ω_2^2 are the two solutions of the quadratic equation,

$$[\omega^2]^2 - \left(\frac{k_1 g}{W_1} + \frac{k_2 g}{W_2} + \frac{k_1 g}{W_2}\right)[\omega^2] + \frac{k_1 k_2 g^2}{W_1 W_2} = 0 \quad . \quad . \quad (3)$$

(and $f_1 = \frac{\omega_1}{2\pi}$ and $f_2 = \frac{\omega_2}{2\pi}$ are the two resonant frequencies).

There are many possible varieties of this system, with various combinations of damping and disturbing force. The theory of the system is important because it represents, in simplified form, (*a*) a spring-borne car on pneumatic tires, (*b*) a railroad car truck with both journal and bolster springs, (*c*) a steam locomotive on a flexible rail, (*d*) a spring-mounted engine in a car, as well as a variety of other practically important systems. Therefore, it is worth giving a general method of calculating the behavior of this system, particularly as similar methods will be used later for other purposes.

At the extreme point of a steady undamped vibration, all the weights are stationary for a moment and each has an acceleration $y\omega^2$ where y is the displacement from equilibrium

ω is the angular velocity corresponding to the harmonic motion, $= 2\pi \times$ frequency.

The system will be in equilibrium under the spring and inertia forces.

For example, in the simple system in Fig. 41

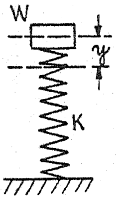


FIG. 41

$$\text{Force} = \text{mass} \times \text{acceleration}$$

$$ky = \frac{W}{g}\, y\, \omega_c^2$$

$$\therefore \quad \omega_c = \sqrt{\frac{kg}{W}}$$

which gives the natural period

$$f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{kg}{W}}$$

as given in Chapter 2.

If the weight is acted on by a force $F \cos \omega t$, then at the extreme position, the force is F and

$$ky - F = \frac{W}{g}\, y\omega^2$$

$$\therefore \quad y = \frac{F}{k - \dfrac{W\omega^2}{g}} = \frac{F}{k} \times \frac{1}{1 - \dfrac{W}{kg}\,\omega^2} = \frac{F}{k} \times \frac{1}{1 - \dfrac{\omega^2}{\omega_c^2}}$$

as given in Chapter 3.

Now applying this to the system of two weights, Fig. 42,

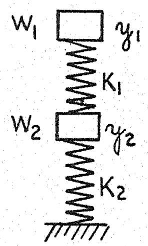


FIG. 42

For the upper weight,

$$k_1(y_1 - y_2) = \frac{W_1}{g}\, y_1\omega_c^2$$

For the lower weight

$$k_2y_2 - \frac{W_1y_1}{g}\,\omega_c^2 = \frac{W_2}{g}\, y_2\omega_c^2$$

Eliminate y_1 and y_2 and these give

$$\left(k_1 - \frac{W_1}{g}\,\omega_c^2\right)\left(k_2 - \frac{W_2}{g}\,\omega_c^2\right) - K_1\,\frac{W_1}{g}\,\omega_c^2 = 0$$

or

$$[\omega_c^2]^2 - \left(\frac{k_1 g}{W_1} + \frac{k_2 g}{W_2} + \frac{k_1 g}{W_2}\right)[\omega_c^2] + \frac{k_1 k_2 g^2}{W_1 W_2} = 0$$

which was given above as equation (3) for finding the natural periods of vibration.

For systems with damping, the weights will not all vibrate together in phase and it is necessary to use vectors to represent the motion of each part.

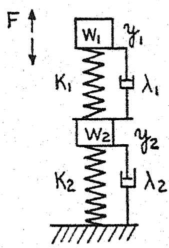


FIG. 43

Example. The system in Fig. 43 has damping

$\lambda_1 \times$ the relative velocity of W_1 and W_2

$\lambda_2 \times$ the relative velocity of W_2 and ground.

There is a force $F \cos \omega t$ acting on W_1.
The equations, force = mass $\times$ acceleration, are:

For W_1

$$F - k_1(y_1 - y_2) - \lambda_1\omega(y_1 - y_2) = -\frac{W_1}{g}\,\omega^2 y_1$$

For W_2

$$k_1(y_1 - y_2) + \lambda_1\omega(y_1 - y_2) - k_2 y_2 - \lambda_2\omega y_2 = -\frac{W_2}{g}\,\omega^2 y_2.$$

These are vector equations, the velocity terms, those with λ, being 90° ahead of the others (Chapter 2). These equations may be rewritten to group the y_1 and y_2 terms.

$$F = \left(k_1 + \lambda_1\omega - \frac{W_1}{g}\,\omega^2\right)y_1 - (k_1 + \lambda_1\omega)y_2 \quad . \quad . \quad (4)$$

and $$(k_1 + \lambda_1\omega)y_1 = (k_1 + \lambda_1\omega)y_2 + \left(k_2 + \lambda_2\omega - \frac{W_2}{g}\,\omega^2\right)y_2. \quad (5)$$

It will be seen that F, y_1 and y_2 are all proportional to each other. It is most convenient to take a unit value of y_2 and obtain y_1 from equation (5). Then F can be obtained from equation (4).

To do this, draw a vector OA (Fig. 44), representing any assumed value of y_2.

In equation (5), the term

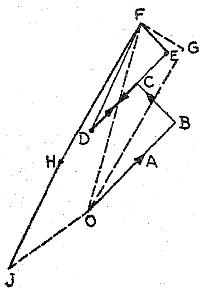


FIG. 44

$$\left(k_2 + \lambda_2\omega - \frac{W_2}{g}\,\omega^2\right) y_2$$

is

$$\begin{aligned} & OB \ (= k_2 OA) \\ & + BC \ (= \lambda_2 \omega OA, \ 90^\circ \text{ ahead of } OA) \\ & + CD \left(= \frac{W_2}{g}\,\omega^2 OA, \ 180^\circ \text{ ahead of } OA \right) \\ & = OD. \end{aligned}$$

The term $(k_1 + \lambda_1\omega)y_2$

is

$$\begin{aligned} & DE \ (= k_1 OA) \\ & + EF \ (= \lambda_1 \omega OA, \ 90^\circ \text{ ahead of } OA). \end{aligned}$$

Hence each side of equation (5) is represented by the vector OF. Now if we draw the triangle OGF similar to the triangle DEF (make the angles equal), OG will represent $k_1 y_1$ and GF will represent $\lambda_1 \omega y_1$.

Now proceed to equation (4).

The term $\left(k_1 + \lambda_1\omega - \frac{W_1}{g}\,\omega^2\right) y_1$

is $OF \ (= [k_1 + \lambda_1\omega] y_1)$

$$+FH \left(= \frac{W_1}{g}\,\omega^2 y_1 = \frac{W_1}{g}\,\omega^2 \times \frac{OF}{\sqrt{k_1^2 + \lambda_1^2\omega^2}}, \ 180^\circ \text{ ahead of } OG. \right.$$

The term $-(k_1 + \lambda_1\omega)y_2$

is $HJ \ (= -FD)$.

Hence, finally, the force F is represented by the vector OJ. From this diagram, given any value of F, any other force or motion can be obtained, together with its phase.

This method may appear laborious when compared with some algebraic methods but it has two advantages. First, when the calculation is done we have the numerical values of all forces and motions obtainable at once from the diagram. If the equations were solved in the shape of a formula it would still be necessary to make numerical calculations which are often much more laborious than the entire graphical solution. Second, the graphical method is general and can be applied where algebraic solutions are difficult. It may be added that if it is necessary to calculate a natural period of a complicated system it is often easiest to make two or three vector diagrams for different assumed values of ω and to find by trial and error what value of ω gives zero disturbing force. Then this gives the natural period and the relative values of all motions can be obtained from the diagram.

We will now return to the system shown in Fig. 36 (b), a weight supported on two springs. First, in the simple case where everything is symmetrical and the weight is equally divided between two similar springs, the nature of the vibration is easy to see. There are two simple ways in which the system can vibrate. First, the weight can vibrate up and down, with the two springs acting together just as if they were one spring; a second simple oscillation occurs when the center of the weight remains steady and one spring moves up while the other moves down, so that the weight see-saws about its center.

The up-and-down motion has a frequency:

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{g}{a}}$$

where g = acceleration due to gravity

a = normal static deflection of the springs.

The see-saw motion has a frequency:

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{g}{a}} \times \frac{l}{k}$$

where $2l$ = distance between the centers of the springs.

k = radius of gyration of the weight about its center.

Any natural vibration is made up of a combination of these two types of vibration. Forced oscillations of both types can occur together. For example, if the system rolls over a wavy road as in Fig. 45 (a) where the distance between two crests is equal to the distance between springs, it will be clear that only the up-and-down motion will be sustained. If the crests are spaced twice the distance between springs, as in Fig. 45 (b), only the see-saw motion will be sustained. For other spacings of the waves in the road, combinations of both vibrations will be sustained.

When the system is not symmetrical there will still be two simple natural types of vibration and all natural vibrations will be made up

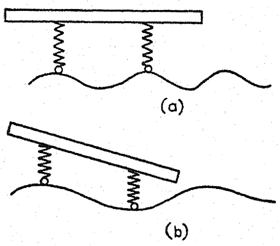


FIG. 45

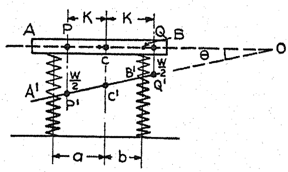


FIG. 46

of these two simple kinds. Each simple kind consists of a vibration in which one point of the weight remains fixed, as though it were pivoted.

Let the body AB in Fig. 46 have a weight W and radius of gyration k, so that it is equivalent to two weights $\frac{W}{2}$ spaced a distance k on each side of the center of gravity C. Let the springs be spaced a and b from the center of gravity and let their stiffnesses (lb. per inch deflection) be S_a and S_b.

Let the weight perform a simple vibration, so that every point moves in a regular harmonic motion and let $A'B'$ be the extreme position of one swing, and AB the static equilibrium position. Let $A'B'$ intersect AB at O, so that the point O acts as if it were pivoted during this simple vibration. Let the angle $AOA' = \theta$.

Then
$$AA' = OA\theta = (OC + a)\theta$$
$$PP' = OP\theta = (OC + k)\theta$$
$$QQ' = OQ\theta = (OC - k)\theta$$
$$BB' = OB\theta = (OC - b)\theta$$

The upward spring forces are:

$$S_a\, AA' \text{ at } A'$$
$$S_b\, BB' \text{ at } B'$$

The downward inertia forces (=mass × acceleration) are:

$$\frac{W}{2g}\,\omega^2\, PP' \text{ at } P'$$
$$\frac{W}{2g}\,\omega^2\, QQ' \text{ at } Q'$$

Where ω = angular velocity corresponding to the motion = 2π × frequency.

Since these must balance, we have, taking moments about O,

$$S_aAA' \times OA + S_bBB' \times OB = \frac{W}{2g}\,\omega^2PP' \times OP + \frac{W}{2g}\,\omega^2QQ' \times OQ$$

$$\therefore\; S_a(OC + a)^2 + S_b(OC - b)^2$$
$$= \frac{W}{2g}\,\omega^2\,[(OC + k)^2 + (OC - k)^2]. \quad (6)$$

Also, equating the forces,

$$S_aAA' + S_bBB' = \frac{W}{2g}\,\omega^2PP' + \frac{W}{2g}\,\omega^2QQ'$$

$$\therefore\; S_a(OC + a) + S_b(OC - b) = \frac{W}{2g}\,\omega^2\,[(OC + k) + (OC - k)]. \quad (7)$$

Eliminating $\frac{W}{2g}\omega^2$ from these two equations, we obtain

$$[S_a(OC + a)^2 + S_b(OC - b)^2 \times 2OC$$
$$= [S_a(OC + a) + S_b(OC - b) \times 2\,[OC^2 + k^2]$$
$$\therefore\ OC^2\,[aS_a - bS_b]$$
$$+ OC\,[(a^2 - k^2)S_a + (b^2 - k^2)S_b] - k^2\,[aS_a - bS_b] = 0$$

Let $$p = \frac{[(a^2 - k^2)S_a + (b^2 - k^2)S_b]}{2\,[aS_a - bS_b]}$$

Then $$OC = -\,p \pm \sqrt{p^2 + k^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Also, from equation (7).

$$\omega^2 = \frac{g}{W}\left[S_a\left(1 + \frac{a}{OC}\right) + S_b\left(1 - \frac{b}{OC}\right)\right] \quad . \quad . \quad . \quad (9)$$

Equation (8) gives the two "pivot points" corresponding to the two simple types of vibration and equation (9) gives the angular velocity corresponding to each type of vibration and hence the frequencies $\left(= \frac{\omega}{2\pi}\right)$. Forced vibrations of both kinds can occur simultaneously and therefore there will be two resonant frequencies, corresponding to the two kinds of vibration.

Equation (8) may be written

$$\frac{OC}{k} = \frac{-p}{k} \pm \sqrt{\left(\frac{p}{k}\right)^2 + 1}$$

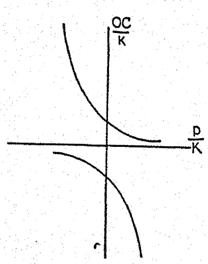


FIG. 47

and this is shown in Fig. 47. It is worth considering some special cases because of their importance. First, if $as_a - bs_b = 0$, which is easily seen to be the condition that both supporting springs have the same static deflection under the load. Then p becomes infinite and the pivot points are at zero and infinity. That is, the oscillations consist of

(*a*) oscillations about the center of gravity, for which

$$\omega^2 = \frac{g}{W} \cdot \frac{S_a a^2 + S_b b^2}{k^2} \qquad \text{from (6)}$$

(b) oscillations straight up and down, for which

$$\omega^2 = \frac{g}{W}(S_a + S_b) \qquad \text{from (9)}$$

The second case of interest occurs when

$$ab = k^2$$

Substituting this for k^2 it can easily be shown that $OC = -a$ or $+b$; that is, that each support is a center of oscillation and the two springs oscillate independently of each other.

With A as a center of oscillation,

$$\omega^2 = \frac{g}{W} \cdot S_b \cdot \left(\frac{a+b}{a}\right)$$

which corresponds to a spring of stiffness S_b loaded with a weight

$$W\left(\frac{a}{a+b}\right).$$

Similarly, with B as a center of oscillation, the vibration of A corresponds to

$$\omega^2 = \frac{g}{W} \cdot S_a\left(\frac{a+b}{b}\right).$$

This relationship, $ab = k^2$, is approached in many modern automobiles and is responsible for some of the comfortable riding. Suppose the car reaches a hole in the road. The front springs start an oscillation as they pass over it. If the springs affect each other, the rear springs will start to oscillate at the same time. When the rear wheels reach the hole, a moment later the effect of the hole on the rear springs may be increased or decreased due to the oscillation started by the front springs, depending on the speed, the size of the hole, etc. The only way in which it can be ensured that the front springs will never affect the rear springs adversely is to make them independent of each other: this is done by designing the car so that $ab = k^2$; that is, each spring is a center of oscillation for the other.

Chapter 5

SYSTEMS WITH SPRING CONSTANTS WHICH VARY WITH TIME

The two examples of interest are the vibration of a vehicle running over a rail whose joints are more flexible than the solid rail and the vibrations in the side-rods of certain kinds of electric locomotives. Let us consider one weight vibrating on one spring and let the stiffness of the spring vary regularly as shown in Fig. 48.

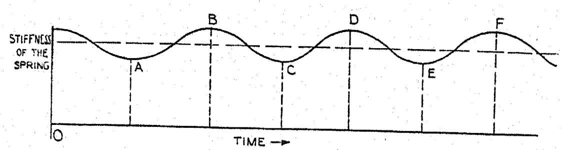


FIG. 48

If, for example, we have a weight running on a rail A, C, E, etc., would represent the moments when the weight passed over the joints, which are more flexible than the solid rail. At very slow speed the weight will obviously follow the shape of the loaded rail and will vibrate in a form very similar to $ABCDE$. There will be resonance when the natural period of the weight corresponds to the time taken to go from one joint to the next, i.e., from A to C.

So far, the behavior of the weight is not very different from what it would be if the rail were stiff and with a wavy surface. However, on the flexible rail there is also resonance at twice the above speed; that is, when the natural period corresponds to the time taken to go from A to E. In addition to this the resonance is not confined to one exact speed but extends over a small range of speed, so that the vibration will increase indefinitely at any speed in this range.

At the speed at which this resonance begins the weight will be

down at, say, A and E and will be up at C, that is, the greatest displacement occurs where the stiffness is least.

Also at B and D, where the stiffness is greatest, the weight is passing through its equilibrium position and the stiffness has no effect. The resultant stiffness is therefore lower than the average and the speed at which resonance starts is lower than the resonant speed corresponding to the average stiffness of the rail. As the speed increases the vibration lags, as in all cases of resonance, until the weight is down at, say, B and F and up at D. It will be seen that the maximum stiffness is now controlling and the speed is higher than that corresponding to resonance with average stiffness. There is clearly resonance at all speeds between these two limits. If the stiffness varies approximately according to a cosine curve, the limits of resonance correspond approximately to stiffnesses half-way between the average and the minimum and maximum. For example, if the stiffness varies from 8 to 10 the stiffnesses for which resonance should be calculated are $8\frac{1}{2}$ and $9\frac{1}{2}$.

Example. A weight rests on a spring with a static deflection of $1\frac{1}{2}$ in. This system runs over a rail whose deflection is $\frac{1}{8}$ in. between joints and $\frac{3}{16}$ in. near joints. The joints are $16\frac{1}{2}$ ft. apart. At what speeds will there be resonance? The average static deflection is $1\frac{1}{2} + \frac{5}{32} = 1.65625$ in. $= 0.13802$ ft.

The natural frequency of vibration is

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{a}} = \frac{1}{6.28}\sqrt{\frac{32.2}{.138}} = 2.43 \text{ per sec.}$$

There will be resonance with every joint at $2.43 \times 16.5 = 40.1$ ft. per sec. $= 27.4$ m.p.h. There will also be resonance with every other joint at twice this speed; that is, $54\frac{3}{4}$ m.p.h. The range of resonance at the higher speed is calculated approximately as follows:

Maximum static deflection	= 1.6875 in.
Minimum " "	= 1.6250 in.
Average " "	= 1.65625 in.
$\frac{1}{4}$(max.-min.) = .0625/4	= 0.015625 in.

Static deflection for lower end of resonant range = 1.671875 in.
Static deflection for upper end of resonant range = 1.640625 in.

The corresponding speeds are $54\frac{1}{2}$ and 55.0 m.p.h. The range is small, as would be expected from the very small fluctuation in stiffness of the spring support. The important thing is the existence of resonance at a speed corresponding to every other joint.

Exact Mathematical Methods. These problems are represented by differential equations with periodic coefficients. In most cases the fluctuation of spring stiffness is reasonably close to a cosine curve. Then, in order to use known formulae the equation should be put in the form

$$\frac{d^2y}{dz^2} + (a + 16q\cos 2z)y = 0.$$

This is called Mathieu's equation. It has periodic solutions, corresponding to the limits of resonance when

$$a = 1 + 8q - 8q^2 - 8q^3 + \cdots$$

and when

$$a = 1 - 8q - 8q^3 + 8q^3 + \cdots$$

In the first case

$$y = \sin z + q\sin 3z + q^2(\sin 3z + \tfrac{1}{3}\sin 5z) + \cdots$$

In the second case

$$y = \cos z + q\cos 3z + q^2(-\cos 3z + \tfrac{1}{3}\cos 5z) + \cdots$$

These are particular cases of a general solution

$$y = e^{\mu z}U$$

where $U = \sin(z - \sigma) + a_3\cos(3z - \sigma) + b_3\sin(3z - \sigma) + \cdots$

in which the angle σ changes from 0 to $\dfrac{\pi}{2}$ in the resonant range between the two periodic solutions. The value of σ is obtained from the equation.

$$a = 1 + 8q\cos 2\sigma + (-16 + 8\cos 4\sigma)q^2 - \cdots$$

and μ is given by

$$\mu = 4q\sin 2\sigma - 12q^3\sin 2\sigma - - \cdots$$

The coefficients a_3 and b_3 are given by:

$$a_3 = 3q^2 \sin 2\sigma + \cdots$$

$$b_3 = q + q^2 \cos 2\sigma + \cdots$$

Near the maximum resonant range, a approaches unity and if the fluctuation in stiffness is small, q is small so that higher powers can be neglected.

Under these special conditions the resonant range is from

$$a = 1 + 8q \text{ to } a = 1 - 8q.$$

At the middle of the range the rate of increase of vibration is greatest and the vibration is given by

$$y = \text{constant} \times e^{4qz} \times \sin\left(z - \frac{\pi}{4}\right).$$

Outside the range of resonance it will be found that cos 2σ is greater than 1, which makes σ complex and sin 2σ and μ imaginary. This type of solution corresponds to the stable forced oscillations outside of the resonant range. For further details and for a solution in which the fluctuations are not simple harmonic, see "On a General Solution of Hill's Equation" by E. Lindsay Ince, *Proceedings Edinburgh Mathematical Society* LXXV, 1915, pp. 436–448.

Chapter 6

HELICAL SPRINGS

A helical spring consists of a bar or wire coiled into the shape of a helix. The common forms of end are illustrated in Fig. 49, in which the upper end is coiled solid and ground flat, so that the last coil is "dead." The lower end is shown bent in to the central axis, which is often convenient for a tension spring.

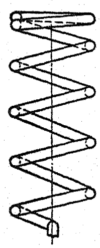


FIG. 49

Vehicle springs are more generally used in compression with flat ends. The following notation will be used:

F = load on the spring
M = moment acting on the spring
R = radius of the center line of the coil
D = diameter of the center line of the coil
r = radius of the cross section of round wire
d = diameter of the cross section of round wire
n = number of active coils
h = active height of the spring, measured at the center line of the wire (center line of spring)
J = moment of inertia of wire section in twist (for round wire) [1]
I = moment of inertia of wire section in bending
G = shear modulus of elasticity of the material
E = tension modulus of elasticity of the material.

The center line of the wire is supposed to lie on a helix $y = c\theta$ so that at one end of the spring $\theta = 0$ and the angle θ increases by 2π for each coil of the wire. It will be noticed that

$$h = 2\pi c \times n$$

[1] For other than round sections, J, the moment of twist is not equal to the polar moment of inertia.

S = deflection of the spring
K = spring constant = load per unit deflection or moment per unit angular twist according to the particular constant concerned.

Stiffness of Helical Springs

Axial Load. The compressive force F at the center of the coil produces a twisting moment $F \times R$ throughout the wire (Fig. 50). This produces an angle of twist of

$$\frac{FR}{GJ} \text{ per unit length of wire (for round wire).}$$

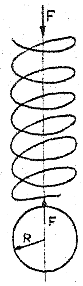


Fig. 50

This twist produces a deflection at the center line, a distance R from the wire of

$$\frac{FR}{GJ} \times R \text{ per unit length of wire.}$$

Therefore the total deflection at the center line is

$$\frac{FR^2}{GJ} \times 2\pi Rn = \frac{2\pi nR^3}{GJ} \times F = \frac{F}{K_1}$$

$$\therefore\ K_1 = \frac{GJ}{2\pi nR^3}$$

For round wire,
$$K_1 = \frac{G \times \left(\dfrac{\pi d^4}{32}\right)}{2\pi n\left(\dfrac{D^3}{8}\right)} = \frac{G}{8n}\frac{d^4}{D^3}$$

Transverse Load. The spring is subject to a transverse force F and the ends are forced to remain parallel. This requires a moment M at each end of the spring, given by

$$M = \frac{F \times h}{2}.$$

Consider any point P on the wire, determined by the angle θ as in Fig. 51 (b).

The force F produces at P, per unit length of wire,

(1) A twist $\dfrac{F\cos\theta \times C\theta}{GJ}$

(2) A bend in the plane of the paper $\dfrac{F \times R\sin\theta}{EI}$

(3) A bend perpendicular to the paper $\dfrac{F\sin\theta \times c\theta}{EI}$

The moment M produces at P, per unit length of wire,

(1) A twist $\dfrac{M\cos\theta}{GJ}$

(2) A bend perpendicular to the paper $\dfrac{M\sin\theta}{EI}$

These produce displacements of the bottom of the spring as follows:

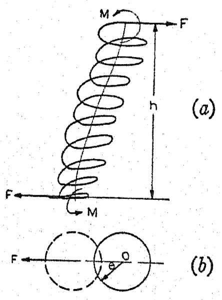


FIG. 51

Due to the force F

(1) $\dfrac{F\cos\theta \times c\theta}{GJ} \times c\theta\cos\theta$

(2) $\dfrac{FR\sin\theta}{EI} \times R\sin\theta$

(3) $\dfrac{F\sin\theta \times c\theta}{EI} \times c\theta\sin\theta.$

Due to the moment M

(1) $\dfrac{-M\cos\theta}{GJ} \times c\theta\cos\theta$

(2) $\dfrac{-M\sin\theta}{EI} \times c\theta\sin\theta.$

In order to find the total displacement of the bottom of the spring we have to take the sum of the effects of every part of the spring from $\theta = 0$ to $\theta = 2\pi n$. Hence, if δ is the deflection,

$$\delta = \int_0^{2\pi n} \left(\frac{Fc^2\theta^2}{GJ} \cos^2 \theta + \frac{FR^2}{EI} \sin^2 \theta \right.$$

$$\left. + \frac{Fc^2}{EI} \theta^2 \sin^2 \theta - \frac{Mc}{GJ} \theta \cos^2 \theta - \frac{Mc}{EI} \theta \sin^2 \theta \right) R d\theta$$

$$= \frac{Fc^2R}{GJ} \left[\frac{(2\pi n)^3}{G} + \frac{2\pi n}{4} \right] + \frac{FR^3}{EI} \left(\frac{2\pi n}{2} \right)$$

$$+ \frac{Fc^2R}{EI} \left[\frac{(2\pi n)^3}{G} - \frac{2\pi n}{4} \right] - \frac{McR}{GJ} \frac{(2\pi n)^2}{4} - \frac{McR}{EI} \frac{(2\pi n)^2}{4}$$

Then, putting $\quad M = \frac{Fh}{2} = \frac{F \times 2\pi nc}{2} = F\pi nc,$

$$\frac{\delta}{F} = \pi nR \left\{ \frac{1}{12} \left(\frac{1}{GJ} + \frac{1}{EI} \right) (2\pi nc)^2 + \frac{R^2}{EI} + \frac{c^2}{2} \left(\frac{1}{GJ} - \frac{1}{EI} \right) \right\}$$

The last term is generally negligible, so that

$$\frac{1}{K_2} = \frac{\delta}{F} = \pi nR \left\{ \frac{1}{12} \left(\frac{1}{GJ} + \frac{1}{EI} \right) (2\pi nc)^2 + \frac{R^2}{EI} \right\}$$

Note that $2\pi nc = h$, the height of the spring.

For round wire,

$$\frac{1}{K_2} = \pi nR \left\{ \frac{h^2}{12} \left(\frac{32}{G\pi d^4} + \frac{64}{E\pi d^4} \right) + \frac{R^2 \times 64}{E\pi d^4} \right\}$$

$$= \frac{8nD}{Ed^4} \left\{ \frac{h^2}{3} \left(\frac{E}{2G} + 1 \right) + D^2 \right\}$$

Axial Couple. A couple M about the axis of the coil bends the wire, producing an angular deflection of

$$\frac{M}{EI} \text{ per unit length.}$$

The total angular deflection is, therefore

$$2\pi nR \times \frac{M}{EI}$$

and the spring constant, moment/angular deflection is

$$K_3 = \frac{EI}{2\pi n R}$$

For round wire,
$$K_3 = \frac{E \times \left(\frac{\pi d^4}{64}\right)}{2\pi n\left(\frac{D}{2}\right)} = \frac{Ed^4}{64nD}$$

Transverse Couple. A transverse couple M has already been investigated above as one of the results of a transverse load. Taking only the terms due to the moment M, we have

$$\delta = \frac{McR}{4}\left(\frac{1}{GJ} + \frac{1}{EI}\right)(2\pi n)^2$$

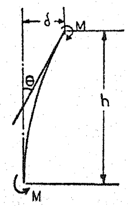


FIG. 52

But, as shown in Fig. 52, if a spring acted on by a pure couple has a deflection at one end of δ, the angular deflection of the center line is

$$\theta = \frac{-2\delta}{h}$$

since the spring center line will be bent into a circular arc.

Therefore, the spring constant K_4 is given by

$$\frac{1}{K_4} = \frac{\theta}{M} = \frac{-2\delta}{Mh} = \frac{cR(2\pi n)^2}{2h}\left(\frac{1}{GJ} + \frac{1}{EI}\right)$$

But $h = 2\pi nc$, therefore

$$\frac{1}{K_4} = \pi n R\left(\frac{1}{GJ} + \frac{1}{EI}\right)$$

For round wire,

$$\frac{1}{K_4} = \pi n\frac{D}{2}\left(\frac{32}{G\pi d^4} + \frac{64}{E\pi d^4}\right) = \frac{32nD}{Ed^4}\left(\frac{E}{2G} + 1\right)$$

Stresses in Helical Springs. Referring to the sections above, which consider stiffness, the shearing forces and bending moments may

be picked out and stresses calculated. For axial load the wire is under a uniform twisting moment $FR = \frac{FD}{2}$.

The section modulus for round wire in twist is $\frac{\pi d^3}{16}$

Hence the maximum shearing stress is $\frac{8FD}{\pi d^3}$

This is a first approximation in which the effect of the curvature of the wire on the distribution of stress is neglected. Referring to the study of a transverse load, it will be found that the maximum shear stress is $\frac{8Fh}{\pi d^3}$ at the ends; and the maximum bending stress, 90° around the helix from the maximum shear stress, is

$$\frac{16F\sqrt{D^2 + h^2}}{\pi d^3}$$

Stresses for other conditions may be calculated similarly.

Stability of Helical Springs. If a long, light spring is compressed it may buckle, just as any strut may buckle if it is too slender for its length and for the compressive load which is put on it. The buckling of struts is covered by theories given in any book on strength of materials.

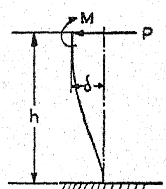


FIG. 53

Consider a bar, Fig. 53, fixed at the lower end and acted on by a lateral force P at the upper end and by a moment M, also at the upper end, which keeps the two ends of the bar parallel.

Let E = modulus of elasticity of the material

I = moment of inertia of section for bending.

Then the deflection of the upper end is

$$\delta = \frac{Ph^3}{12EI}$$

Now consider that the bar represents the center line of a helical spring: we have

$$\delta = \frac{P}{K_2}$$

where K_2 is the lateral spring constant, calculated above. Thus the spring may be expected to behave in this respect like a bar for which

$$EI = \frac{K_2 h^3}{12}$$

The theoretical buckling load of a strut whose ends remain parallel is

$$F = \frac{4\pi^2 EI}{h^2}$$

so that the theoretical compression load which would cause the spring to buckle is

$$F = \frac{\pi^2}{3} hK_2 = \frac{\pi^2 Ed^4}{8nD\left\{h\left(\dfrac{E}{2G} + 1\right) + \dfrac{3D^2}{h}\right\}}$$

This theory may not cover the exact conditions at the ends of the spring and may not allow for other possible ways in which the spring may buckle, so that considerable margin of safety should be allowed when using this formula.

Chapter 7

RUBBER SPRINGS

General. The word rubber covers a wide range of material with properties ranging from soft sponge to hard ebonite. Raw rubber, obtained from plants, is subjected to a variety of processes in the manufacture of the finished rubber used in springs, and of these compounding and vulcanizing are of most interest. The rubber is compounded with materials such as lead salts, zinc oxide, vulcanization accelerators and antioxidants. It is later vulcanized; that is, it is compounded with sulphur or other materials at suitable temperatures. In general the more sulphur that is absorbed the harder is the rubber. Spring rubber often contains about 5–10% of sulphur.

It is not possible to give here any complete or detailed account of the properties of various kinds of rubber but a few factors which are of particular importance will be mentioned briefly.

(1) Rubber is almost incompressible, being comparable with water. If rubber is to compress in one direction it must be allowed to expand in another. Soft rubber resembles a liquid in many ways.

(2) Its modulus of elasticity is low, of the order of 200–1200 lb./sq. in. as compared to 30,000,000 lb./sq. in. for steel.

(3) When placed under a steady load, rubber continues to deform or "drift." The drift is considerable at first and continues at a slower and slower rate. Usually the significant drift takes place in something of the order of 1 hour, but slight drift continues indefinitely.

(4) If rubber is loaded and unloaded it does not behave elastically but takes a temporary set and absorbs some energy.

(5) If unloaded rubber is left free the set will gradually disappear.

(6) If rubber is repeatedly loaded and unloaded it will gradually reach a steady cycle after which it no longer takes a set and in which the energy absorption is less than for a single cycle, unless too much heat is generated.

(7) Rubber is very sensitive to temperature. An increase of temperature generally makes rubber softer and increases the drift.

(8) It is a poor conductor of sound and similar high frequency vibrations.

(9) It is destroyed by oil or by oxidation. Protective coatings, shielding from direct sunlight and protection from excessive heat should result in rubber having a life of 5–10 years in suitable applications.

Rubber is often tested for approximate properties by the Shore Durometer. This indents the surface of the rubber with a needle and measures the force required to produce a given indentation. The scale is arbitrary. The lower the Durometer reading, the softer the rubber.

As a very rough indication of the variation of modulus of elasticity with Durometer hardness the following table may be used, but wide variations will be found in different classes of material:

Shore Durometer Hardness	*Approx. Modulus of Elasticity*
40	275 lb./sq. in.
50	375 " "
60	550 " "
70	750 " "

The shear modulus for rubber may be assumed to be approximately $\frac{1}{3.5}$ times the modulus for direct load.

On account of the great variation in properties of different rubbers it is not practical to use general rules in detail design. Particular cases should be investigated by those experienced in the special materials involved. In order to give a general indication of the order of quantities involved the following rough rules are given, but it must be remembered that they may give results which are far from accurate.

A rough guide to the drift of rubber is taken by considering a standard loading which produces a deflection, measured after 1 min., of 15% of the thickness of the rubber. This may be shear or compression. Then the drift after 24 hours is roughly

$$\frac{E}{50}\,\%$$

and the drift after one year is roughly

$$1.5\sqrt{E} \text{ for shear, or}$$

$$\frac{E}{10} \text{ for compression}$$

where E is in each case the modulus of elasticity for direct load.

As an example, if rubber of compression modulus $E = 550$ is loaded in shear with an initial deflection of 15%, the drift after 24 hours will be

$$\frac{E}{50}\% = \frac{550}{50}\% = 11\%$$

and the deflection will be $(15\%)\left(1 + \frac{11}{100}\right) = 16.6\%$

After one year the drift will be

$$1.5\sqrt{E} = 1.5\sqrt{550} = 35\%$$

and the deflection will be $(15\%)\left(1 + \frac{35}{100}\right) = 20.3\%$

Rubber may be bonded to steel with an ultimate strength of the order of 500 lb./sq. in. Working static pressure often does not exceed 25 lb./sq. in. shear on the bond, although in carefully studied applications shears of 60 lb./sq. in. can be used with success.

In compression the loading on rubber is generally limited to that which produces not over 20% deflection.

Advantages of Rubber. The chief advantages which may in some cases make rubber springs preferable to steel ones are as follows:

(1) It is often easy to combine a rubber spring with a guide so that clearances, friction and wear are eliminated. This is closely related to the ability of rubber to act as a satisfactory spring in two or three directions at once.

(2) Rubber is an efficient insulator of sound and high frequency vibration when used with reasonably large deflection. A mere rubber pad, heavily loaded, is not a proper insulator, but a suitably designed spring is.

(3) Rubber springs absorb energy, but without the friction and

inertia of steel leaf springs. They damp out high frequency vibration effectively. If low frequency vibrations are encountered it may be necessary to add shock absorbers or other damping devices.

Rubber Sandwiches in Compression. Two steel plates separated by rubber which is bonded to the plates form a combination called a rubber sandwich. Calculation of the characteristics of these sandwiches involves various difficulties and the formulae given here are simplified to afford some quick indication of the order of the quantities involved.

Let E = modulus of elasticity of rubber in compression—lb./sq. in.
T = thickness of rubber in inches
Q = a length in inches which depends on the size and shape of the sandwich
F = load on sandwich, lb.
A = area of cross section, sq. in.

Then, approximately, for a unit loading of 150 lb./sq. in. and a value of $\dfrac{Q}{T} = 5$ and a rubber for which $E = 200$ lb./sq. in., we have a deflection of 15%.

Within reasonable limits of this case,

$$\text{Deflection }\% = 15 \times \frac{\left(\dfrac{F}{A}\right)}{150} \times \frac{200}{E} \times \frac{5}{\left(\dfrac{Q}{T}\right)} \quad . \quad . \quad . \quad (10)$$

The formula can of course be simplified to

Deflection % $= \dfrac{100FT}{AEQ}$, but it is well to remember the values for which it is approximately correct and therefore it will be left in the form (10).

The value of Q may be taken as follows:

(*a*) For a square, Q = length of one side
(*b*) For a rectangle, Q = width (the shorter side), Fig. 54 (*a*)

(*c*) For a circle, Q = diameter, Fig. 54 (*b*)
(*d*) For a circle with a hole, $Q = D_1 - D_2$, Fig. 54 (*c*).

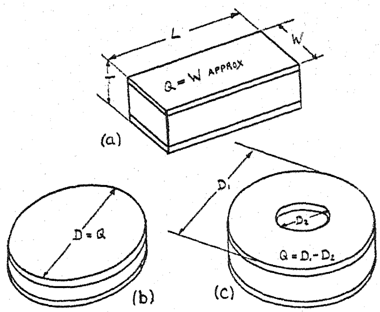


FIG. 54

Example. A rectangular sandwich is 10 × 18 in. with rubber of modulus $E = 300$, and $2\frac{1}{2}$ in. thick. If the deflection is to be limited to 20%, what load can the sandwich carry.

Here $Q = 10$ in.
$T = 2\frac{1}{2}$ in.
$E = 300$ lb./sq. in.
$A = 180$ sq. in.

Therefore $$20 = 15 \times \frac{\left(\frac{F}{180}\right)}{150} \times \frac{200}{300} \times \frac{5}{4}$$

or $$\frac{F}{180} = 240 \text{ lb./sq. in.}$$

or the total load is approximately $180 \times 240 = 43{,}000$ lb.

Rubber Sandwiches in Shear. If a sandwich, such as that shown in Fig. 55 (*a*), is subjected to a shearing force and at the same time the two metal plates are held parallel, the rubber will deform in the way shown in Fig. 55 (*b*). For simplicity the deformed condition is gen-

erally shown as in Fig. 55 (*c*) in which the curves in the rubber are neglected.

If T = thickness of rubber in in.
A = area of cross section of rubber, sq. in.
F = total shearing force in lb.
S = shear modulus of rubber in lb. per sq. in.
θ = angular deflection in deg.
d = deflection in in.

Then

$$\frac{F}{AS} = \frac{\theta^\circ}{57.3} \quad \ldots \ldots \ldots \quad (11)$$

and

$$d = T \tan \theta^\circ.$$

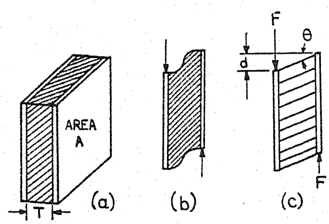


FIG. 55

For small deflections, $\frac{\theta^\circ}{57.3}$ is approximately equal to tan θ so that,

for small deflections, $\frac{F}{AS} = \frac{d}{T}$.

In the majority of cases the deflections will be sufficiently large so that the more accurate formula (11) should be used.

The formula applies to sandwiches of any shape but should be used with caution if the width becomes comparable to the thickness. If the sandwich is under compression or tension as well as shear, the thickness to be used in the formula is the original thickness of the unstressed sandwich.

Circular Sandwich in Torsion.

Let R_0 = radius of sandwich (Fig. 56)
R_1 = radius of hole

T = thickness of rubber
S = shear modulus of rubber
$\theta°$ = angular twist of sandwich
d = deflection of a point on the outer edge.

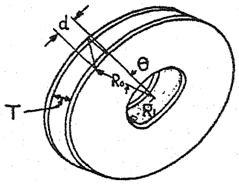


FIG. 56

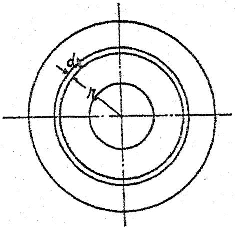


FIG. 57

Consider a small ring of radius r (Fig. 57) and width dr, then, using the formula given in the preceding section for small deflections, the twisting moment due to this small ring is

$$(2\pi r dr) \times S \times \frac{r\theta°}{57.3T} \times r$$

Therefore the total twisting moment is

$$\int_{R_1}^{R_0} \frac{2\pi S\theta}{57.3 \times T} r^3 dr = \frac{\pi S\theta°}{2 \times 57.3T}\left(R_0{}^4 - R_1{}^4\right) = \frac{\pi Sd}{2T}\left(\frac{R_0{}^4 - R_1{}^4}{R_0}\right)$$

and the maximum stress in the rubber is

$$\frac{Sd}{T} \text{ at the outer radius } R_0.$$

It is worth noting that if the twisting moment and the maximum stress are fixed, the sandwich with the least amount of rubber is that for which the ratio $\frac{R_0}{R_1}$ approaches 1. Therefore the most economical size, as far as the amount of rubber is concerned, is a ring whose maximum outside diameter is only limited by design reasons.

This is shown as follows:

The formulae above show that, if the stress and the twisting moment are constant, then

$$\frac{R_0^4 - R_1^4}{R_0} \quad \text{is constant.}$$

Also the weight of rubber in the sandwich is proportional to the cross-sectional area and therefore to $R_0^2 - R_1^2$.

$$\text{Let } x = \frac{R_1}{R_0} \quad \text{and} \quad W = \text{weight of rubber.}$$

Then W is proportional to $R_0^2(1 - x^2)$

$$\text{and } R_0^3(1 - x^4) \quad \text{is constant.}$$

$$\text{Therefore } W \text{ is proportional to } \frac{1 - x^2}{(1 - x^4)^{2/3}}$$

$$\text{This has a minimum value when } \frac{dW}{dx} = 0 \quad \text{that is,}$$

when $\quad (1 - x^4)^{2/3}(-2x) = (1 - x^2)\frac{2}{3}(1 - x^4)^{-1/3}(-4x^3)$

or $\quad x(1 - x^4) = \frac{4}{3}x^3(1 - x^2).$

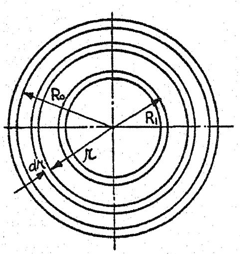


FIG. 58

From this x must be either 0, 1 or $\sqrt{3}$ of which $x = 1$ is the only real value which gives a minimum.

Concentric Tube Spring. Let the thickness of the rubber be T and the axial force between the inner and outer tubes F (Fig. 58). Then on a small ring of radius r and width dr, the shear stress is $\dfrac{F}{2\pi rT}$ and, for small deflections, the deflection is $\dfrac{F\,dr}{2\pi rTS}$

Therefore the total deflection is

$$\int_{R_1}^{R_0} \frac{F}{2\pi TS}\frac{dr}{r} = \frac{F}{2\pi TS}\log_e\left(\frac{R_0}{R_1}\right) = 0.37\,\frac{F}{TS}\log_{10}\left(\frac{R_0}{R_1}\right)$$

The maximum shear stress will be

$$\frac{F}{2\pi R_1 T}$$

at the inner tube.

The same spring may be used to resist a twisting moment between the inner and outer tubes. In this case, if the twisting moment is M, the shear on a small ring is

$$\frac{M}{r} \times \frac{1}{2\pi r T}$$

the deflection is
$$\frac{M}{r} \times \frac{1}{2\pi r T} \times \frac{dr}{S}.$$

At the outer radius this produces a deflection

$$\frac{M}{r} \times \frac{1}{2\pi r T} \times \frac{dr}{S} \times \frac{R_0}{r}$$

and the total deflection is

$$\int_{R_1}^{R_0} \frac{MR_0}{2\pi TS} \frac{dr}{r^3} = \frac{MR_0}{4\pi TS}\left(\frac{1}{R_1^2} - \frac{1}{R_0^2}\right)$$

The maximum shear stress will be

$$\frac{M}{2\pi R_1^2 T}$$

at the inner tube.

For a given stress, moment and volume of rubber, the deflection increases proportionately to the radius.

A useful variation of the concentric tube spring is made with rubber of varying thickness so that the shearing stress will be uniform and the amount of rubber in the spring will therefore be decreased.

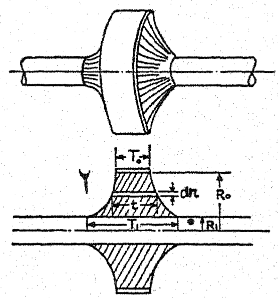


FIG. 59.

A spring of this kind is shown in Fig. 59 and if t is the thickness of the rubber at radius r, then tr is constant to give uniform shear stress

due to axial load. With an axial force F, the stress is

$$\frac{F}{2\pi R_0 T_0} = \frac{F}{2\pi R_1 T_1}$$

The deflection is

$$\frac{F(R_0 - R_1)}{2\pi R_0 T_0 S}$$

For uniform shear due to a twisting moment M, the rubber would be shaped so that tr^2 is constant.

Then the shear stress is $\dfrac{M}{2\pi R_0{}^2 T_0} = \dfrac{M}{2\pi R_1{}^2 T_1}$

The deflection is

$$\int_{R_1}^{R_0} \frac{M}{2\pi R_0{}^2 T_0 S} \frac{R_0}{r}\, dr = \frac{M}{2\pi R_0 T_0 S} \log_e \left(\frac{R_0}{R_1}\right)$$

$$= 0.37 \frac{M}{R_0 T_0 S} \log_{10} \left(\frac{R_0}{R_1}\right)$$

Note on the Design of Rubber Springs. The springs described above and combinations of them are often particularly suitable for use when it is necessary to produce definite flexibilities in all directions. For example, a rubber sandwich may be designed to have suitable flexibility in compression and in shear and may then be given any torsional flexibility required, over a wide range. Similarly a concentric tube spring is generally used to be flexible towards axial motion or shear and stiff against other motions.

It may be well to repeat that the formulae given for rubber springs are rough approximations. They are useful in getting some general idea of the order of quantities concerned and are also useful in prorating from one spring to another which differs only slightly from it. When accurate values are required it is necessary to refer to the results of tests or to the more complicated formulae which apply to certain particular cases which have been studied.

Chapter 8

RIDING COMFORT

A problem closely allied to that of vibration of vehicles is that of the comfort of passengers. This is partly a question of the vibration which is actually transmitted to the passenger and partly affected by other factors such as:

(1) The visual effect of motion of surrounding objects
(2) Distraction of attention by an ever-changing scene
(3) Noise
(4) The exhilarating effect of a ride

Tests have been made both in the laboratory and on moving vehicles to study ride comfort and the results have been interpreted in various ways by different experimenters. Complete discussion of the question will not be included here but there will be given an outline of the basic facts which have been established and one method which has been used with some success to compare different observations and to predict the probable comfort of riders in a vehicle where vibration is known.

Cushion Effect. It is a matter of common observation, confirmed by tests, that a passenger sitting on a cushioned seat can comfortably withstand greater vibrations of his body than if he sits on a hard, unpadded seat. A "comfortable" seat does two things. It reduces the vibration transmitted from a car to a passenger and it also increases the amount of vibration which he can withstand comfortably, probably because his weight is better distributed over a larger area.

On a cushioned seat the maximum acceleration of the passenger's body is a good indication of his comfort. Over a range of frequency of 3 to 12 cycles per sec. the comfortable limit of acceleration is roughly $2\frac{1}{2}$ to 4 ft. per sec. per sec. Accelerations higher than this are disturbing.

On a hard seat the conditions are somewhat similar for low frequencies, of 3 or 4 cycles per sec., but the limit of comfortable acceleration drops very quickly as the frequency increases. At a frequency of 6 cycles per sec., for example, accelerations above about 0.8 ft. per sec. per sec. are disturbing, as compared with about 4 for the cushioned seat.

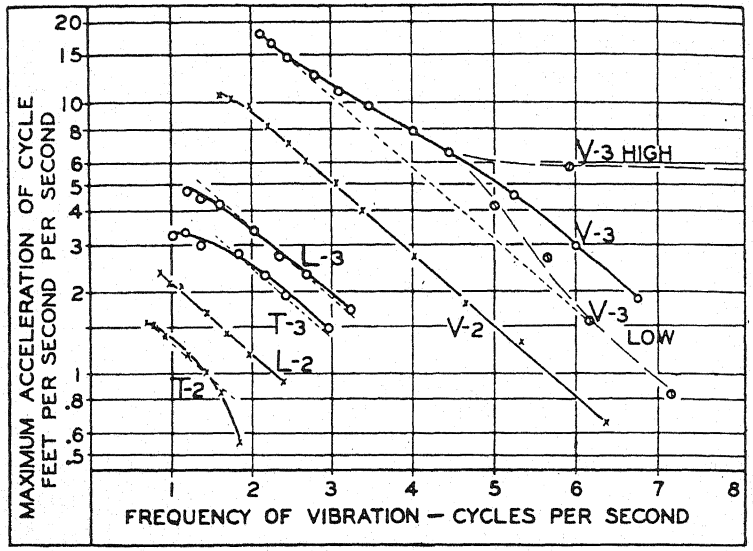


FIG. 60

Comfort Index for Unpadded Seats. Fig. 60 shows acceleration plotted against frequency, representing the result of tests on a vibrating platform at Purdue University. In this graph,

V represents vertical vibration
L represents longitudinal vibration
T represents transverse vibration
2 represents the limit of comfortable acceleration, above which the motion is disturbing.
3 represents the limit of disturbing acceleration, above which the motion is uncomfortable.

The words were defined as follows:

Disturbing—You note that certain organs or parts of your body have greater vibration than you yourself, and you try to prevent this by tightening up on certain muscles.

Uncomfortable—You now want *very little* of the treatment.

It will be noticed that the curve of test points for uncomfortable vertical vibration has two branches. Two definite groups of people were found, one group more sensitive than the other. In fitting a simple curve to the points so as to obtain a comfort index the more sensitive group was taken. Vibrations greater than this will be uncomfortable to some persons, although it is lower than the average for all.

It was found at Purdue that the results could be represented reasonably by a formula

$$K = Ae^{0.6f}$$

where K is a constant called the *comfort index*.

A is the maximum acceleration in ft. per sec. per sec.

e is the constant 2.7183

f is the frequency of vibration in cycles per sec.

The following limits were obtained:

Event	*Direction*	*Maximum "K"*
Uncomfortable	Vertical	64.7
Disturbing	Vertical	31.2
Uncomfortable	Longitudinal	11.73
Disturbing	Longitudinal	4.02
Uncomfortable	Transverse	8.21
Disturbing	Transverse	2.35

The most convenient way to calculate the comfort index is by the use of the nomogram in Fig. 61.

Acceleration is marked on the left scale and frequency on the right scale. A straight line drawn between them will give the comfort index on scale K. For example, if the acceleration is 8 ft./sec./sec. and the frequency is 1 cycle per sec., the diagram shows that the comfort index is 14, which would be very uncomfortable if it were in

a horizontal direction but not at all disturbing if it were in a vertical direction.

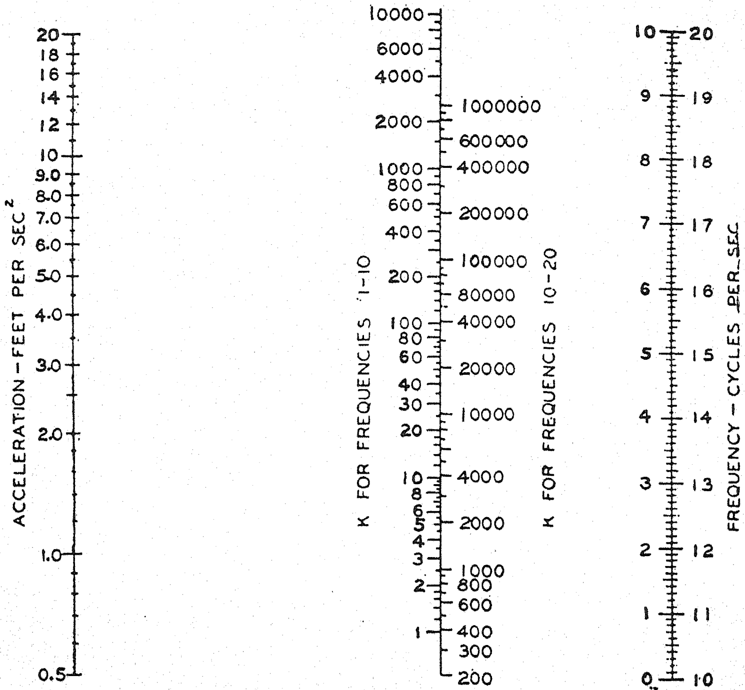


FIG. 61. NOMOGRAPH FROM WHICH VALUES OF K FOR HARD SEATS MAY BE OBTAINED

A straight edge connecting the values for acceleration and frequency will intersect the center reference line at the proper value for K.

Combined Directions. A method was worked out at Purdue University to give an index for simultaneous vibration in two or three directions.

First, the longitudinal and transverse comfort indices are calculated and combined by the formula

$$\text{Horizontal comfort index} = \sqrt{K_L^2 + K_T^2}$$

where K_L is the comfort index for longitudinal motion
K_T is the comfort index for transverse motion

This horizontal index is then combined with the index for vertical motion on the chart, Fig. 62.

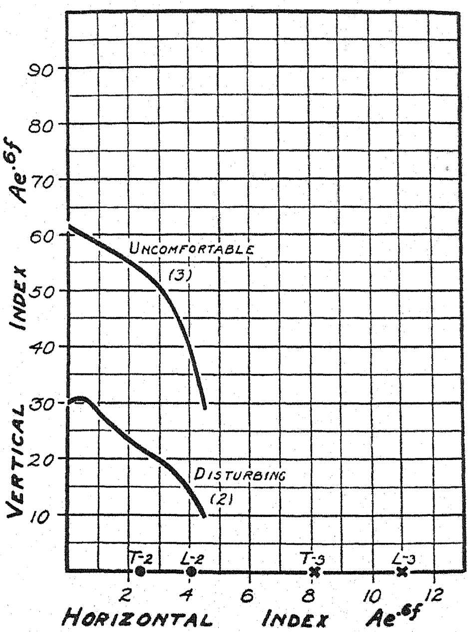


FIG. 62. CHART FOR SHOWING INDICES COMBINED TO GIVE BORDERLINE REACTIONS

For example, suppose we have the following combination of vibrations:

Vertical—2 ft./sec./sec. max. acceleration at 3 cycles/sec.
Longitudinal—0.4 ft./sec./sec. max. acceleration at 3 cycles/sec.
Transverse—1 ft./sec./sec. max. acceleration at 2 cycles/sec.

By using the nomogram (Fig. 61) the comfort indices are:

$$K_V = 12$$

$$K_L = 2.5$$

$$K_T = 3.3$$

The combined horizontal index is:

$$K_H = \sqrt{K_L^2 + K_T^2} = \sqrt{2.5^2 + 3.3^2} = 4.1$$

If the point corresponding to a vertical index of 12 and a horizontal index of 4.1 is plotted in Fig. 62 it will be seen that it lies just under the limit of comfortable vibration and any increase would result in crossing the "disturbing" line.

Comfort Index for Cushioned Seats. As a result of automobile tests at Purdue with instruments placed on cushioned seats, relations between maximum acceleration, frequency and comfort were obtained which were reduced approximately to the following values:

Event	*Direction*	*Expression*
Uncomfortable	Vertical	$K_V = 10 = Ae^{0.13f}$
Disturbing	Vertical	$K_V = 8.5 = Ae^{0.13f}$
Uncomfortable	Longitudinal	$K_L = 5.5 = Ae^{0.087f}$
Disturbing	Longitudinal	$K_L = 4 = Ae^{0.087f}$
Uncomfortable	Transverse	$K_T = A_T = 3.25$
Disturbing	Transverse	$K_T = A_T = 2.75$

Combining these to give the vector sum, K_C, of the forces at zero frequency as follows:

$$K_C = \sqrt{K_V^2 + K_L^2 + K_T^2}$$

it is found that:

$$K_C \text{ (Uncomfortable)} = \sqrt{10^2 + 5.5^2 + 3.25^2} = 11.9 \text{ ft. per sec.}^2$$

$$K_C \text{ (Disturbing)} = \sqrt{8.5^2 + 4^2 + 2.75^2} = 9.78 \text{ ft. per sec.}^2$$

The nomographic chart, Fig. 63, enables K_V and K_L to be calculated easily for any values of acceleration and frequency.

For a more complete discussion of these results, reference should be made to "Human Reactions to Vibration" by H. M. Jacklin, *Transactions S.A.E.*, 1936, Vol. 31, pp. 401–407.

Variation of Results. Some indication has already been given that the results of comfort tests vary from one individual to another, and that of course is what we would expect. It is, therefore, interesting to know whether this variation between individuals is great enough to make general formulae useless in practice.

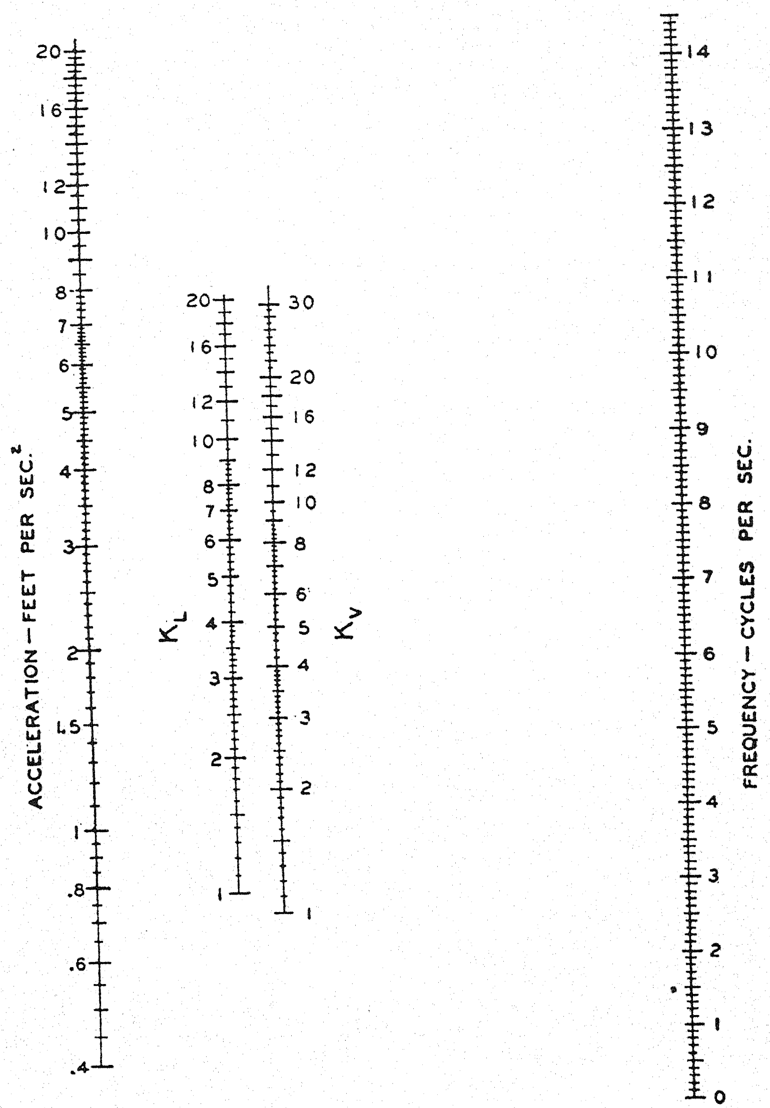

ACCELERATION — FEET PER SEC.²
20
16
12
10
8
7
6
5
4
3
2
1.5
1
.8
.6
.4
K_L
20
16
12
10
8
7
6
5
4
3
2
1
K_V
30
20
16
12
10
8
6
5
4
3
2
1
FREQUENCY — CYCLES PER SEC.
14
13
12
11
10
9
8
7
6
5
4
3
2
1
0

FIG. 63

Fig. 64 shows the variation of 93 observations on the Purdue vibrating platform.

The vibration was kept at a total amplitude of $\frac{1}{20}$ in. and the subjects decided at what frequency it became "disturbing." The mean is about 280 cycles and 67 out of the 93 readings fall within $\pm 20\%$ of this value. This represents a normal chart and indicates

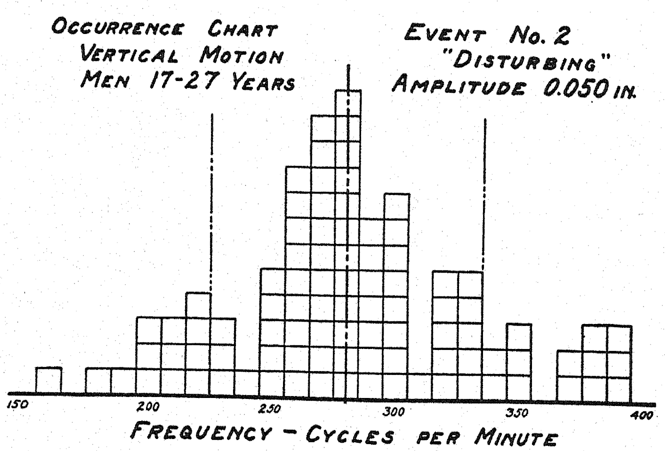


FIG. 64

sufficiently close grouping of the observations to make the results of value.

Application of the Results. In various sections of this book the amplitude or maximum acceleration of some vibration is calculated. The formulae given in this section permit this to be converted to comfort index. For example in Fig. 65 the dotted line shows the simplest relation between amplitude of vibration and frequency of a vibrating car. The light full line shows maximum acceleration, or $4\pi^2 f^2 \times$ frequency. The heavy full line shows comfort index, or $e^{0.6f} \times$ acceleration. Similar procedure is applicable to more complicated cases.

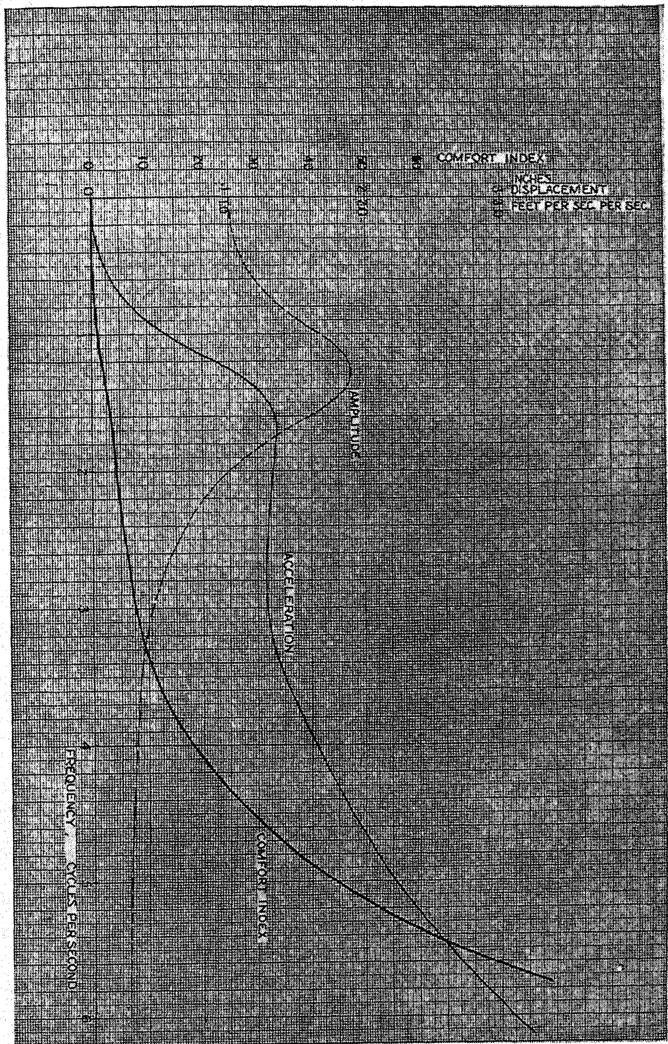


FIG. 65. RIDING COMFORT (1)

Chapter 9

SOLUTION OF DIFFERENTIAL EQUATIONS

The method which is usually most convenient for solving the equations of motion of vibrating systems will be outlined here. The method will be developed by merely stating the rules in a form which can be used without much study. Readers who are interested in the theory should refer to other works in which Bromwich's contour integrals are discussed.* The method used here is based on the work of Heaviside. It is particularly suited to getting a complete solution directly when the conditions under which the vibration starts are known. We will use two special symbols;

p will be used to represent the operation of differentiation, $\frac{d}{dx}$, and will be treated like any algebraic quantity.

1 will be used to represent a function which is zero until the motion starts and unity thereafter.

One Degree of Freedom. The typical equation of such a system without external force is

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + \omega^2 y = 0$$

Writing p for $\frac{d}{dx}$, this becomes

$$(p^2 + bp + \omega^2)y = 0$$

If, at the instant from which we measure the time, the displacement is U and the velocity is v, we write an auxiliary equation,

$$(p^2 + bp + \omega^2)y = (p^2 + bp)U + pv \quad . \quad . \quad . \quad (12)$$

* See, for example, *Mathematics of Modern Engineering* by Doherty, R. E., and Keller, E. G., John Wiley & Sons, Inc., 1936.

The right-hand side of this auxiliary equation is obtained by substituting $y = U + \frac{v}{p}$ in the left-hand side and disregarding constant terms and negative powers of p in the result. That is

$$(p^2 + bp + \omega^2)\left(U + \frac{v}{p}\right) = (p^2 + bp)U + pv + \text{constants and terms}$$

$$\text{in } \frac{1}{p} \text{ and } \frac{1}{p^2}, \text{ which are disregarded.}$$

We now solve the auxiliary equation (12) for y in the usual way and obtain

$$y = \frac{p^2 + bp}{p^2 + bp + \omega^2} U + \frac{p}{p^2 + bp + \omega^2} v \quad . \quad . \quad . \quad (13)$$

There are rules, which will be given more fully later, for interpreting expressions like

$$\frac{p^2 + bp}{p^2 + bp + \omega^2} \quad \text{and} \quad \frac{p}{p^2 + bp + \omega^2}$$

and these give, directly,

$$y = Ue^{-\frac{b}{2}t}\left[\cos\sqrt{\left(\omega^2 - \frac{b^2}{4}\right)}t + \frac{b \sin}{2\sqrt{\omega^2 - \frac{b^2}{4}}}\sqrt{\left(\omega^2 - \frac{b^2}{4}\right)}t\right]$$

$$+ \frac{ve^{-\frac{bt}{2}}}{\sqrt{\omega^2 - \frac{b^2}{4}}} \sin\sqrt{\left(\omega^2 - \frac{b^2}{4}\right)}t$$

If some external force P (per unit mass) starts to act on the system at the moment from which we measure the time, let the system be at rest in the equilibrium position until the time when the force P starts to act. The force is $P\mathit{1}$ and the equation of motion is

$$(p^2 + bp + \omega^2)y = P\mathit{1}$$

Now, just as most functions of p can be interpreted as functions of

time, so most forces which occur in vibration work can be expressed as functions of p, for example if the force is

$$P = F \sin nt$$

the rules allow us to write this as

$$P\mathit{1} = F \frac{np}{p^2 + n^2} \mathit{1}$$

from which

$$y = \frac{P\mathit{1}}{p^2 + bp + \omega^2} = \frac{Fnp\mathit{1}}{(p^2 + bp + \omega^2)(p^2 + n^2)} \quad . \quad . \quad (14)$$

This is interpreted by the rules as

$$y = \frac{F}{\sqrt{(\omega^2 - n^2)^2 + b^2n^2}} \sin(nt - \delta_1) - \frac{Fn}{\sqrt{\omega^2 - \frac{b^2}{4}}} e^{-\frac{b}{2}t}.$$
$$\sin\left(\sqrt{\omega^2 - \frac{b^2}{4}}\, t - \delta_2\right)$$

where $\tan \delta_1 = \dfrac{bn}{\omega^2 - n^2}$, $\tan \delta_2 = \dfrac{b\sqrt{\omega^2 - \frac{b^2}{4}}}{\omega^2 - \frac{b^2}{2} - n^2}$

The rules which allow us to obtain these results are:

$$\frac{p}{p + \alpha} = e^{-\alpha t}, \quad \frac{p}{p - \alpha} = e^{\alpha t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{1}{p + \alpha} = \frac{1}{\alpha}(1 - e^{-\alpha t}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\frac{1}{p} = t, \quad \frac{1}{p^n} = \frac{t^n}{n!} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$\frac{1}{(p + \alpha)(p + \beta)} = \frac{1}{\alpha\beta} + \frac{1}{\alpha - \beta}\left[\frac{e^{-\alpha t}}{\alpha} - \frac{e^{-\beta t}}{\beta}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{p}{(p + \alpha)(p + \beta)} = \frac{1}{\alpha - \beta}[e^{-\beta t} - e^{-\alpha t}] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$\frac{p^2}{(p+\alpha)(p+\beta)} = \frac{1}{\alpha-\beta}\left[\alpha e^{-\alpha t} - \beta e^{-\beta t}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$\frac{1}{(p+\alpha)^2} = \frac{1}{\alpha^2}\left[1 - e^{-\alpha t}(1+\alpha t)\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\frac{p}{(p+\alpha)^2} = te^{-\alpha t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\frac{p^2}{(p+\alpha)^2} = e^{-\alpha t}(1-\alpha t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\left.\begin{aligned} \frac{1}{(p+\beta)^2+\omega^2} &= \frac{1}{\omega^2+\beta^2}\left[1 - \frac{\sqrt{\omega^2+\beta^2}}{\omega} e^{-\beta t}\sin(\omega t+\varphi)\right] && 9(a) \\ &\text{where } \tan\varphi = \frac{\omega}{\beta} \\ \frac{1}{p^2+\omega^2} &= \frac{1}{\omega^2}(1-\cos\omega t) && 9(b) \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{p\omega}{(p+\beta)^2+\omega^2} &= e^{-\beta t}\sin\omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . && 10(a) \\ \frac{p\omega}{p^2+\omega^2} &= \sin\omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . && 10(b) \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{p(p+\beta)}{(p+\beta)^2+\omega^2} &= e^{-\beta t}\cos\omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . && 11(a) \\ \frac{p^2}{p^2+\omega^2} &= \cos\omega t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . && 11(b) \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{p\omega\cos\varphi \pm p(p+\beta)\sin\varphi}{(p+\beta)^2+\omega^2} &= e^{-\beta t}\sin(\omega t \pm \varphi) \quad . \quad . \quad . \quad . \quad . && 12(a) \\ \frac{p(p+\beta)\cos\varphi \mp \omega p\sin\varphi}{(p+\beta)^2+\omega^2} &= e^{-\beta t}\cos(\omega t \pm \varphi) \quad . \quad . \quad . \quad . \quad . && 12(b) \end{aligned}\right\}$$

Heaviside's expansion theorem, of which all the above are particular cases, is

$$\frac{f(p)}{F(p)} = \frac{f(0)}{F(0)} + \Sigma_\alpha \frac{f(\alpha)}{\alpha F'(\alpha)} e^{\alpha t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where $f(p)$ and $F(p)$ are functions of p

α is a root of the equation $F(p) = 0$

Σ represents the sum of the terms obtained by giving α successively the values of all the roots and $F'(\alpha) = \dfrac{d}{dp}[F(p)]$ when $p = \alpha$.

There is an alternative method of dealing with an external force $P\mathit{1}$. Suppose the equation is

$$(p^2 + bp + \omega^2)y = P\mathit{1} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

First solve
$$(p^2 + bp + \omega^2)y = \mathit{1}$$

and let the solution be
$$y = A(t)$$

where $A(t)$ is some function of t. Also write $P = f(t)$.

Then the solution of equation (15) is

$$y = f(0)A(t) + \int_0^t A(t - \lambda)f'(\lambda)d\lambda$$

which is Duhamel's superposition integral.

The unit function $\mathit{1}$ is often omitted where no confusion will result.

Example. Referring to equation (13)

$$y = \frac{p^2 + bp}{p^2 + bp + \omega^2}U + \frac{p}{p^2 + bp + \omega^2}V$$

an examination of the rules given above shows that this can be written in the forms contained in rules 11 (a) and 10 (a), thus:

$$\frac{p^2 + bp}{p^2 + bp + \omega^2} = \frac{p\left(p + \frac{b}{2}\right)}{\left(p + \frac{b}{2}\right)^2 + \left(\omega^2 - \frac{b^2}{4}\right)} + \frac{p\frac{b}{2}}{\left(p + \frac{b}{2}\right)^2 + \left(\omega^2 - \frac{b^2}{4}\right)}$$

$$= e^{-\frac{b}{2}t}\cos\sqrt{\omega^2 - \frac{b^2}{4}}\,t + \frac{b}{2\sqrt{\omega^2 - \frac{b^2}{4}}}e^{-\frac{b}{2}t}$$

$$\sin\sqrt{\omega^2 - \frac{b^2}{4}}\,t$$

and $\dfrac{p}{p^2 + bp + \omega^2}$ is interpreted directly by rule 10(a)

as
$$\frac{1}{\sqrt{\omega^2 - \frac{b^2}{4}}} e^{-\frac{b}{2}t} \sin \sqrt{\omega^2 - \frac{b^2}{4}}\, t$$

thus giving the solution in the text.

Example. Referring to equation (14)

$$y = \frac{Fnp}{(p^2 + bp + \omega^2)(p^2 + n^2)}$$

which may be interpreted by Heaviside's expansion theorem, rule 13. Here $(p^2 + bp + \omega^2)(p^2 + n^2) = 0$ has the roots

$$p = -\frac{b}{2} \pm i\sqrt{\omega^2 - \frac{b^2}{4}} \quad \text{or} \quad = \pm\, in \text{ where } i = \sqrt{-1}$$

Therefore, substituting these values of α in rule 13,

$$y = Fn \left[\frac{e^{\left[-\frac{b}{2} + i\sqrt{\left(\omega^2 - \frac{b^2}{4}\right)}\right]t}}{2i\sqrt{\omega^2 - \frac{b^2}{4}}\left[\frac{b^2}{2} - \omega^2 + n^2 - ib\sqrt{\left(\omega^2 - \frac{b^2}{4}\right)}\right]} \right.$$
$$+ \frac{e^{\left[-\frac{b}{2} - i\sqrt{\omega^2 - \frac{b^2}{4}}\right]t}}{-2i\sqrt{\omega^2 - \frac{b^2}{4}}\left(\frac{b^2}{2} - \omega^2 + n^2 + ib\sqrt{\omega^2 - \frac{b^2}{4}}\right)}$$
$$\left. + \frac{e^{int}}{(\omega^2 - n^2 + ibn)2in} + \frac{e^{-int}}{(\omega^2 - n^2 - ibn) \times (-2in)} \right]$$

$$= Fn \left[\frac{\left\{ \begin{array}{l} e^{-\frac{b}{2}t}\left\{ b\sqrt{\omega^2 - \frac{b^2}{4}} \cos\sqrt{\omega^2 - \frac{b^2}{4}}\, t \right. \\ \left. \quad + \left(\frac{b^2}{2} - \omega^2 + n^2\right) \sin\sqrt{\omega^2 - \frac{b^2}{4}}\, t \right\} \end{array} \right\}}{\sqrt{\omega^2 - \frac{b^2}{4}}\left\{\omega^4 + n^4 + n^2(b^2 - 2\omega^2)\right\}} \right.$$

$$+\frac{-bn\cos nt+(\omega^2-n^2)\sin nt}{n\{(\omega^2-n^2)^2+b^2n^2\}}\Bigg]$$

$$=\frac{F\sin(nt-\delta_1)}{\sqrt{(\omega^2-n^2)^2+b^2n^2}}$$

$$-\frac{Fn}{\sqrt{\omega^2-\dfrac{b^2}{4}}}\,\frac{e^{-\frac{b}{2}t}\sin\left[\sqrt{\omega^2-\dfrac{b^2}{4}}\,t-\delta_2\right]}{\sqrt{\omega^4+n^4+n^2(b^2-2\omega^2)}}$$

where $$\tan\delta_1=\frac{bn}{\omega^2-n^2}$$

$$\tan\delta_2=\frac{b\sqrt{\omega^2-\dfrac{b^2}{4}}}{\omega^2-\dfrac{b^2}{2}-n^2}$$

Example. Referring to the end of Chapter 2, let a spring-supported weight roll over a single low spot in the road. (See Fig. 66.)

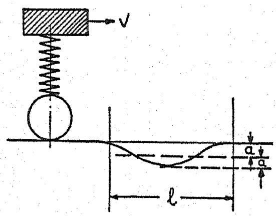


FIG. 66

Let $\omega_c = 2\pi \times$ natural frequency of vibration of the weight on the spring

$2a$ = depth of low spot

l = length of low spot

v = speed at which the weight rolls

y = downward motion of the weight below the original equilibrium position.

$$\omega=\frac{2\pi v}{l}$$

Then $$\frac{d^2y}{dt^2}+\omega_c^2(y-a\{1-\cos\omega t\})=0$$

or $$\frac{d^2y}{dt^2}+\omega_c^2y=\omega_c^2a(1-\cos\omega t)$$

or $$(p^2 + \omega_c^2)y = \omega_c^2 a \times \frac{\omega^2}{p^2 + \omega^2} \quad \text{from rule 9 }(b)$$

$$y = \frac{a\omega_c^2\omega^2}{(p^2 + \omega_c^2)(p^2 + \omega^2)}$$

$$= \left(\frac{1}{p^2 + \omega_c^2} - \frac{1}{p^2 + \omega^2}\right) \times \frac{a\omega_c^2\omega^2}{\omega^2 - \omega_c^2}$$

$$= \frac{a}{\omega^2 - \omega_c^2}[\omega^2(1 - \cos \omega_c t) - \omega_c^2(1 - \cos \omega t)]$$

from $9(b)$

$$= a - a\left(\frac{\omega^2 \cos \omega_c t - \omega_c^2 \cos \omega t}{\omega^2 - \omega_c^2}\right)$$

$$= a - a\left[\frac{\cos \omega t - \dfrac{\omega^2}{\omega_c^2}\cos \omega_c t}{1 - \dfrac{\omega^2}{\omega_c^2}}\right]$$

which applies from $t = 0$ to $t = \dfrac{l}{v}$

when $t = \dfrac{l}{v}$, $$y = a - a\left[\frac{1 - \dfrac{\omega^2}{\omega_c^2}\cos 2\pi\dfrac{\omega_c}{\omega}}{1 - \dfrac{\omega^2}{\omega_c^2}}\right]$$

$$\frac{dy}{dt} = -a\left[\frac{\dfrac{\omega^2}{\omega_c^2}\sin 2\pi\dfrac{\omega_c}{\omega}}{1 - \dfrac{\omega^2}{\omega_c^2}}\right]$$

After the weight has passed over the low spot, the equation of motion is

$$\frac{d^2y}{dT^2} + \omega_c^2 y = 0$$

and we will now measure the time, T, from the moment the weight passes the end of the low spot. Then the auxiliary equation is, from (12)

$$(p^2 + \omega_c^2)y = p^2U + pv$$

$$\therefore\ y = \frac{p^2}{p^2 + \omega_c^2}\,U + \frac{p}{p^2 + \omega_c^2}\,V$$

$$= U \cos \omega_c t + \frac{v}{\omega_c} \sin \omega_c t \quad \text{from 11 } (b) \text{ and 10 } (b)$$

where
$$U = a\left[1 + \frac{-1 + \dfrac{\omega^2}{\omega_c^2}\cos 2\pi\dfrac{\omega_c}{\omega}}{1 - \dfrac{\omega^2}{\omega_c^2}}\right]$$

$$v = -\,a\omega_c\left[\frac{\dfrac{\omega^2}{\omega_c^2}\sin 2\pi\dfrac{\omega_c}{\omega}}{1 - \dfrac{\omega^2}{\omega_c^2}}\right]$$

Several Degrees of Freedom. Equations with several variables are handled in just the same way as above, for example if the equations are:

$$(a_{11}p^2 + b_{11}p + c_{11})y_1 + (a_{12}p^2 + b_{12}p + c_{12})y_2 = S_1$$
$$(a_{21}p^2 + b_{21}p + c_{21})y_1 + (a_{22}p^2 + b_{22}p + c_{22})y_2 = S_2$$

the auxiliary equations are:

$$(a_{11}p^2 + b_{11}p + c_{11})y_1 + (a_{12}p^2 + b_{12}p + c_{12})y_2 =$$
$$(a_{11}p^2 + b_{11}p)U_1 + a_{11}pv_1 + (a_{12}p^2 + b_{12}p)U_2 + a_{12}pv_2 + S_1$$
$$(a_{21}p^2 + b_{21}p + c_{21})y_1 + (a_{22}p^2 + b_{22}p + c_{22})y_2 =$$
$$(a_{21}p^2 + b_{21}p)U_1 + a_{21}pv_1 + (a_{22}p^2 + b_{22}p)U_2 + a_{22}pv_2 + S_2$$

These simultaneous equations are solved for y_1, and y_2 and the resulting functions of p are interpreted according to the rules given above.

It may be worth a reminder that equations of this kind are most easily solved by using determinants.*

If in the equations

$$\begin{array}{l} a_{11}x_1 + \cdots + a_{1n}x_n = k_1 \\ \cdots \qquad\quad \cdots \\ a_{ni}x_i + \cdots + a_{nn}x_n = k_n \end{array}$$

the determinant

$$\Delta = \begin{vmatrix} a_{11} \cdots a_{1n} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} \cdots a_{nn} \end{vmatrix} \quad \text{is not equal to zero,}$$

the equations have exactly one solution, namely

$$x_1 = \frac{\begin{vmatrix} k_1 a_{12} \cdots a_{1n} \\ k_2 a_{22} \cdots a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ k_n a_{n2} \cdots a_{nn} \end{vmatrix}}{\Delta}, \quad x_2 = \frac{\begin{vmatrix} a_{11} k_1 a_{13} \cdots a_{1n} \\ a_{21} k_2 a_{23} \cdots a_{2n} \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} k_n a_{n3} \cdots a_{nn} \end{vmatrix}}{\Delta}, \cdots$$

$$x_n = \frac{\begin{vmatrix} a_{11} a_{12} \cdots k_1 \\ a_{21} a_{22} \cdots k_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{n1} a_{n2} \cdots k_n \end{vmatrix}}{\Delta}$$

* See Mathematics of Modern Engineering, Doherty and Keller, Vol. 1, p. 63.

Note on the Rules for Interpreting Functions of p. It is possible to prove that

$$f(p)\mathit{1} = \frac{1}{2\pi i}\int_c \frac{f(\lambda)e^{\lambda t}}{\lambda}\,d\lambda$$

where C is either a closed curve enclosing all the roots of $\lambda\Delta(\lambda) = 0$ or a line from $-i\infty$ to $+i\infty$ of such form that all singularities of the integrand are on the left. Readers familiar with contour integration will already be familiar with this formula and others who may wish to study the subject are referred to works on mathematics. The formula is merely given here for convenience of reference, together with the reminder that

$$\int_c f(z)dz = 2\pi i \times \text{the sum of the residues}$$

where C encloses the poles or isolated singularities at which the residues are calculated and the residue of $f(z)$ at the pole or singularity a is the coefficient of $\dfrac{1}{z - a}$ in the Laurent expansion.

It will be noted that the symbol $\mathit{1}$ is generally omitted wherever the omission does not cause confusion.

Solution of Cubic Equations. The cubic equation results from undamped systems with three degrees of freedom. If the equation is divided by the coefficient of x^3 it will be of the form

$$x^3 + a_2x^2 + a_1x + a_0 = 0$$

The second term can always be eliminated by substituting

$$x = y - \frac{a_2}{3}$$

Then $$y^3 + \left(a_1 - \frac{a_2}{3}\right)y + \left(a_0 - \frac{a_1a_2}{3} + \frac{2}{27}a_2{}^3\right) = 0$$

or $$y^3 + ay + b = 0$$

We have reduced the coefficients to two, and will now reduce them to one.

Multiply by $\dfrac{a^3}{b^3}$ and the equation becomes

$$\left(\frac{ay}{b}\right)^3 + \frac{a^3}{b^2}\left(\frac{ay}{b}\right) + \frac{a^3}{b^2} = 0$$

or, putting
$$z = \frac{ay}{b}, \quad c = \frac{a^3}{b^2}$$

$$z^3 + C(z + 1) = 0$$

The real roots of this equation can be obtained from a graph of Z against C, which is easily drawn from the equation,

$$C = \frac{-z^3}{1 + z}$$

For a complex root, $p + iq$ substitute this in, and we have

$$p^3 + 3ip^2q - 3pq^2 - iq^3 + c(p + iq + 1) = 0$$

Both real and imaginary parts must be zero, so that

$$p^3 - 3pq^2 + c(p + 1) = 0 \quad . \quad . \quad . \quad . \quad . \quad (16)$$

and
$$3p^2q - q^3 + cq = 0 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

From the latter, if q is not zero, that is, if the root is not real,

$$q^2 = 3p^2 + C \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and substituting this in (16)

$$p^3 - 3p(3p^2 + c) + c(p + 1) = 0$$

or
$$-8p^3 - 2pc + c = 0$$

$$\therefore \ c = \frac{8p^3}{1 - 2p} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

From (19) and (18) c can be plotted against p and q.

A complete graph is shown in Fig. 67 which can be used to solve any cubic equation. If preferred, either p or q can be obtained from the curves and the equation (18) used to calculate the other.

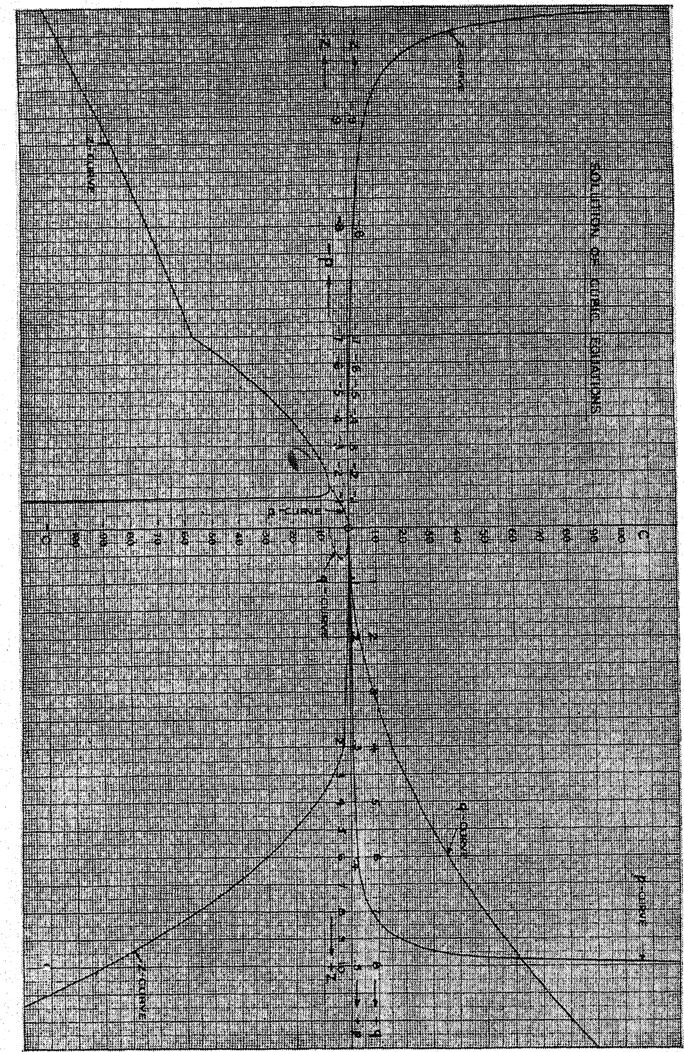

SOLUTION OF CUBIC EQUATIONS
p-CURVE
q-CURVE
Z-CURVE
p-CURVE
q-CURVE
Z-CURVE

FIG. 67

Example. Find the roots of $x^3 + 4x^2 + 14x + 20 = 0$

Let $$x = y - \tfrac{4}{3}$$

Then $$y^3 + 8.67y + 6.07 = 0$$

Let $$z = \frac{8.67}{6.07}y, \quad c = \frac{8.67^3}{6.07^2} = 17.68$$

From Fig. 66, $Z = -0.951$

Therefore $x\left(= \frac{6.07}{8.67}z - \frac{4}{3}\right) = -2 \quad \text{or} \quad -1 \pm 3i$

Solution of Fourth Power Equations. These arise from damped systems of two degrees of freedom or from undamped systems of four degrees of freedom. The method of solution is similar to that given above for cubics, but instead of the coefficients being reduced to one, they can be reduced only to two. The solution is obtained from a set of curves instead of from a single curve.

Dividing by the coefficient of x^4, the equation will be of the form

$$x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

Remove the second term by substituting $x = y - \frac{a_3}{4}$ and the equation becomes

$$y^4 + dy^2 + ay + b = 0$$

Multiply by $\frac{a^4}{b^4}$, put $\frac{ay}{b} = z, \quad \frac{a^4}{b^3} = c, \quad \frac{da^2}{b^2} = f$

Then $$z^4 + fz^2 + c(z + 1) = 0$$

This can be plotted from the equation

$$c = \frac{-z^4 - fz^2}{1 + z}$$

which gives a series of curves for different values of the coefficient f.

For a complex root $p + iq$

$$p^4 + fp^2 - 6p^2q^2 + cp + q^4 - fq^2 + c = 0$$

and $$4p^3q - 4pq^3 + 2fpq + cq = 0$$

From the latter $$q^2 = \frac{4p^3 + 2fp + c}{4p}$$

and by substitution, $c = -\ 8p^2 \pm 2p\sqrt{(f + 4p^2)^2 + 16p^2}$

and $$q^2 = p^2 - 2p + \frac{f}{2} \pm \frac{1}{2}\sqrt{(f + 4p^2)^2 + 16p^2}$$

Graeffe's Method of Solving Equations—Introduction. This method is of great value because it gives all the roots of any algebraic equation as accurately as may be necessary.

The underlying principle of Graeffe's method is readily explained by means of a quadratic equation whose roots are real and distinct.

Consider the equation

$$x^2 + 10.1x + 1 = 0$$

Suppose its roots ($x_2 = -\ 10, x_1 = -\ \frac{1}{10}$) are unknown. There exists a simple routine method whereby we can transform

$$x^2 + 10.1x + 1 = 0$$

into an equation whose roots will be some high even power (say the 256th) of the roots of $x^2 + 10.1x + 1 = 0$. (It is, of course, not necessary to know the roots of the original equation to do this.) The roots of the derived equation are then found easily, as may be seen by the following example. Let the derived equation be

$$x^2 - a_1x + a_2 = 0$$

where a_1 and a_2 are known by the above-mentioned routine process which will be described later. Since the roots of the last equation are $x_1{}^{256}$ and $x_2{}^{256}$ we have

$$x^2 - a_1x + a_2 = (x - x_1{}^{256})(x - x_2{}^{256})$$

$$= x^2 - (x_1{}^{256} + x_2{}^{256})x + (x_1x_2)^{256} = 0$$

Hence $x_1{}^{256} + x_2{}^{256} = a_1$ . . . . (20)

and $(x_1x_2)^{256} = a_2$ . . . . (21)

Let x_2 designate the root with the larger absolute value. Since x_2 is numerically greater than x_1 we can neglect $x_1{}^{256}$ in equation (20) and

solve for x_2. [That is, in the example, $(-\frac{1}{10})^{256}$ is surely negligible in comparison with $(-10)^{256}$.] Since x_2 is now known, x_1 is easily determined from (21).*

Graeffe's Method of Solving Equations—Routine. The method is divided into four parts:

(*a*) Construct a table
(*b*) Determine where to stop the table
(*c*) Calculate the numerical values of the roots
(*d*) Substitute the roots in the equation to find the sign of the roots and to check the calculation

The table is for the purpose of obtaining equations whose roots are the 2nd, 4th, 8th, etc., power of the roots of the original equation. The coefficients of the final equation are used for calculation of the roots.

If the roots are real and unequal, repetition of this root-squaring process ultimately gives coefficients which are the squares of previous ones.

If two real roots are equal, say the (m)th and $(m+1)$th, then the coefficient of x^{n-m} in successive equations is only half the expected value.

If the equation has pairs of complex roots, the number of coefficients which fluctuate in sign is equal to the number of pairs of complex roots. The $(2s)$th power of the modulus is the quotient of the coefficients on either side of the irregular coefficient, if s is the power to which the roots have been raised.

The procedure will be illustrated by solving the equation

$$x^7 - 6x^6 + 25x^5 - 78x^4 + 164x^3 - 224x^2 + 180x - 72 = 0$$

First make a column for each power of x and in the first row put the various coefficients of each power of x_1 thus

	x^7	x^6	x^5	x^4	x^3	x^2	x^1	x^0
1st	1	−6	+25	−78	+164	−224	+180	−72

* Ibid., p. 98. This may be referred to for a more detailed treatment of Graeffe's method.

Underneath each coefficient write:

The coefficient squared
− Twice the product of the coefficients on each side
+ Twice the product of the next two coefficients on each side
− etc.

For example, in the column under x^5, write

$$
\begin{array}{ccccc}
x^7 & x^6 & x^5 & x^4 & x^3 \\
1 & -6 & +25 & -78 & +164 \\
\hline
 & & 25^2 = 625 & & \\
\multicolumn{5}{c}{(-2) \times (-6) \times (-78) = -936} \\
\multicolumn{5}{c}{(+2) \times 1 \times (+164) = \underline{+328}} \\
 & & & & 17
\end{array}
$$

These terms are added and the sums are placed in the row labeled 2nd.

The process is repeated and the successive rows are labeled 1st, 2nd, 4th, 8th, 16th, 32nd, 64th, etc., each number being twice the preceding one. This number refers to the power to which the original roots of the equation have been raised.

The complete table is shown in Table I.

As the number of rows is increased, the coefficients may behave in any of four different ways:

(1) The coefficient is the square of the corresponding coefficient in the previous row. (It is then called regular.)

(2) The coefficient is $\frac{1}{2}$, $\frac{1}{3}$ or some other part of the square of the corresponding coefficient in the previous row.

(3) The coefficient fluctuates in sign without approaching the relations in (1) or (2).

(4) The coefficient follows no obvious law.

Stop the tabulation when the coefficients in classes (1) and (2) approach the squares or fractions of the squares of the previous coefficients to a sufficient degree of accuracy. The tabulation stops with the nth row.

For example in Table I it is stopped with the 32nd row because the coefficients of x^5 and x^0 in the 32nd row are the squares of those of the 16th row.

The three coefficients of x^4, x^3 and x^1 are fluctuating in sign. Therefore there will be three pairs of complex roots and only one real root. The coefficients of x^6 and x^2 follow no obvious law. The three adjacent coefficients of x^3, x^2 and x^1 are irregular, the outside two of which (x^3 and x^1) fluctuate in sign, which means that there will be one

TABLE I

	x^7	x^6	x^5	x^4	x^3	x^2	x^1	x^0
1st	1	-6	$+2.5 \times 10$	-7.8×10	$+1.64 \times 10^{2}$	-2.24×10^{2}	$+1.80 \times 10^{2}$	-7.2×10
	1	3.6×10 -5.0×10	$+6.25 \times 10^{2}$ -9.36×10^{2} $+3.28 \times 10^{2}$	$+6.084 \times 10^{3}$ -8.20×10^{3} $+2.688 \times 10^{3}$ -3.600×10^{2}	$+2.6896 \times 10^{4}$ -3.4944×10^{4} $+9.000 \times 10^{3}$ -8.64×10^{2}	$+5.0176 \times 10^{4}$ -5.904×10^{4} $+1.1232 \times 10^{4}$	$+3.24 \times 10^{4}$ -3.2256×10^{4}	$+5.184 \times 10^{3}$
2nd	1	-1.4×10	$+1.7 \times 10$	$+2.12 \times 10^{2}$	$+8.8 \times 10$	$+2.368 \times 10^{3}$	$+1.44 \times 10^{2}$	$+5.184 \times 10^{3}$
	1	$+1.96 \times 10^{2}$ -3.4×10	$+2.89 \times 10^{2}$ $+5.936 \times 10^{3}$ $+1.76 \times 10^{2}$	$+4.4944 \times 10^{4}$ -2.992×10^{3} -6.6304×10^{4} -2.88×10^{2}	$+7.744 \times 10^{3}$ -1.004×10^{6} $+4.896 \times 10^{3}$ $+1.4515 \times 10^{5}$	$+5.6074 \times 10^{6}$ -2.5344×10^{4} $+2.1980 \times 10^{6}$	$+2.0736 \times 10^{4}$ -2.4551×10^{7}	$+2.6874 \times 10^{7}$
4th	1	$+1.62 \times 10^{2}$	$+6.401 \times 10^{3}$	-2.464×10^{4}	-8.4621×10^{5}	$+7.7801 \times 10^{6}$	-2.453×10^{7}	$+2.6874 \times 10^{7}$
	1	$+2.6244 \times 10^{4}$ -1.2802×10^{4}	$+4.0973 \times 10^{7}$ $+7.9834 \times 10^{6}$ -1.6924×10^{6}	$+6.0713 \times 10^{8}$ $+1.0833 \times 10^{10}$ $+2.5207 \times 10^{9}$ $+4.906 \times 10^{7}$	$+7.1608 \times 10^{11}$ $+3.8340 \times 10^{11}$ -3.1403×10^{11} $+8.7072 \times 10^{9}$	$+6.0530 \times 10^{13}$ -4.1515×10^{13} -1.3244×10^{12}	$+6.0172 \times 10^{14}$ -4.1816×10^{14}	$+7.2221 \times 10^{14}$
8th	1	$+1.3442 \times 10^{4}$	$+4.7264 \times 10^{7}$	$+1.4010 \times 10^{10}$	$+7.7674 \times 10^{11}$	$+1.7691 \times 10^{13}$	$+1.8356 \times 10^{14}$	$+7.2221 \times 10^{14}$
	1	$+1.8069 \times 10^{8}$ -9.4528×10^{7}	$+2.2339 \times 10^{15}$ -3.7664×10^{14} $+1.5535 \times 10^{12}$	$+1.9628 \times 10^{20}$ -7.3424×10^{19} $+4.7561 \times 10^{17}$ -3.6712×10^{14}	$+6.0332 \times 10^{23}$ -4.9570×10^{23} $+1.7352 \times 10^{22}$ -1.9416×10^{19}	$+3.1297 \times 10^{26}$ -2.8516×10^{26} $+2.0236 \times 10^{25}$	$+3.3694 \times 10^{28}$ -2.5553×10^{28}	$+52.1587 \times 10^{28}$
16th	1	$+8.6162 \times 10^{7}$	$+1.8588 \times 10^{15}$	$+1.2331 \times 10^{20}$	$+1.2495 \times 10^{23}$	$+4.8046 \times 10^{25}$	$+8.141 \times 10^{27}$	$+52.1587 \times 10^{28}$
	1	$+7.4239 \times 10^{15}$ -3.7176×10^{15}	$+3.4551 \times 10^{30}$ -2.1249×10^{28} $+2.4990 \times 10^{23}$	$+1.5205 \times 10^{40}$ -4.6451×10^{38} $+8.27948 \times 10^{33}$ -1.6282×10^{28}	$+1.56125 \times 10^{46}$ -1.18491×10^{46} $+3.0265 \times 10^{43}$ -8.9882×10^{37}	$+2.3084 \times 10^{51}$ -2.0344×10^{51} $+1.2863 \times 10^{50}$	$+6.6276 \times 10^{55}$ -5.0120×10^{55}	$+2.7205 \times 10^{59}$
32nd	1	$+3.7063 \times 10^{15}$	$+3.4339 \times 10^{30}$	$+1.4740 \times 10^{40}$	$+3.7937 \times 10^{45}$	$+4.0263 \times 10^{50}$	$+1.6156 \times 10^{55}$	$+2.7205 \times 10^{59}$

set of complex *double* roots. The 4 mth power of the modulus r_2 of the double roots is equal to the quotient of the coefficients on either side of the group of three irregular coefficients.

$$\text{Therefore } (r_2{}^4)^{32} = \frac{\text{coefficient of } x^0}{\text{coefficient of } x^4} = \frac{2.7205 \times 10^{59}}{1.474 \times 10^{40}} = 1.8457 \times 10^{19}$$

$$r_2{}^2 = 2.000$$

The real and imaginary parts of the complex roots and the real root are then obtained through the following relations: As found above the roots of the given equation

$$x^7 + a_1x^6 + a_2x^5 + a_3x^4 + a_4x^3 + a_5 x^2 + a_6x + a_7 = 0 \quad . \quad . \quad . (1)$$

are $$x_1, r_1e^{\pm i\phi_1}, \quad r_2e^{\pm i\phi_2}$$

The equation whose roots are the mth power of these roots is:

$$(x + x_1{}^m)(x + r_2{}^m e^{im\phi_2})^2(x + r_2{}^m e^{-im\phi_2})^2(x + r_1 e^{im\phi_2})(x + r_1{}^m e^{-m\phi_2}) = 0$$

or x^7

$$+x^6[4r_2{}^{2m} \cos m\phi_2 + 2r_1{}^m \cos m\phi_1 + x_1{}^m]$$

$$+x^5[2r_2{}^{2m} + 4r_2{}^{2m} \cos^2 m\phi_2 + 8r_1{}^m r_2{}^m \cos m\phi_1 \cos m\phi_2 + r_1{}^{2m} + x_1{}^m(4r_2{}^m \cos m\phi_2 + 2r_1{}^m \cos m\phi_1)]$$

$$+x^4[4r_2{}^m r_2{}^{2m} \cos m\phi_2 + 4r_1{}^m r_2{}^{2m} \cos m\phi_1 + 8r_1{}^m r_2{}^{2m} \cos m\phi_1 \cos^2 m\phi_2 + 4r_1{}^{2m} r_2{}^m \cos m\phi_2 + x_1{}^m(2r_2{}^{2m} + 4r_2{}^{2m} \cos^2 m\phi_2 + 8r_1{}^m r_2{}^m \cos m\phi_1 \cos m\phi_2 + r_1{}^{2m})]$$

$$+x^3[r_2{}^{4m} + 8r_1{}^m r_2{}^{3m} \cos m\phi_1 \cos m\phi_2 + 2r_1{}^{2m} r_2{}^{2m} + 4r_1{}^{2m} r_2{}^{2m} \cos^2 n\phi_2 + x_1{}^m(4r_2{}^m r_2{}^{2m} \cos m\phi_2 + 4r_1{}^m r_2{}^{2m} \cos m\phi_1 + 8r_1{}^m r_2{}^{2m} \cos m\phi_1 \cos^2 m\phi_2 + 4r_1{}^{2m} r_2{}^m \cos m\phi_2)]$$

$$+x^2[2r_1{}^m r_2{}^{4m} \cos m\phi_1 + 4r_1{}^{2m} r_2{}^{3m} \cos m\phi_2 + x_1{}^m(r_2{}^{4m} + 8r_1{}^m r_2{}^{3m} \cos m\phi_1 \cos m\phi_2 + 2r_1{}^{2m} r_2{}^{2m} + 4r_1{}^{2m} r_2{}^{2m} \cos^2 m\phi_2)]$$

$$+ x^1[r_1{}^{2m} r_2{}^{4m} + x_1{}^m(2r_1{}^m r_2{}^{4m} \cos m\phi_1 + 4r_1{}^{2m} r_2{}^{3m} \cos m\phi_2)]$$

$$+x^0[x_1{}^m r_1{}^{2m} r_2{}^{4m}] = 0$$

Assuming that the roots arranged in descending order are

$$r_1e^{\pm i\phi_1}, \quad x_1, \quad (r_2e^{\pm i\phi_2})^2$$

we obtain the following dominant equation:

$$x^7$$
$$+ x^6[2r_1{}^m \cos m\phi_1] + x^5[r_1{}^{2m}]$$
$$+x^4[x_1{}^m r_1{}^{2m}] + x^3[x_1{}^m 4r_1{}^{2m} r_2{}^m \cos m\phi_2]$$
$$+x^2[2r_1{}^{2m} x_1{}^m r_2{}^{2m}(1 + 2 \cos^2 m\phi_2)$$
$$+x[4r_1{}^{2m} x_1{}^m r_2{}^{3m} \cos m\phi_2]$$
$$+x_1{}^m r_1{}^{2m} r_2{}^{4m}$$

If we apply the root squaring process once more to the dominant equation we find:

$$
\begin{aligned}
& x^7 \\
& +x^6[4r_1{}^{2m} \cos^2 m\phi_1 \\
& \qquad -2r_1{}^{2m}] \\
& +x^5[r_1{}^{4m} \\
& \qquad -4r_1{}^{3m} \cos m\phi_1 x_1{}^m \\
& \qquad +8r_1{}^{2m}r_2{}^m x_1{}^m \cos m\phi_2] \\
& +x^4[x_1{}^{2m}r_1{}^{4m} \\
& \qquad -2 \times 4x_1{}^m r_1{}^{4m} r_2{}^m \cos m\phi_2 \\
& \qquad +8r_1{}^{3m}x_1{}^m r_2{}^{2m} \cos m\phi_1(1 + 2\cos^2 m\phi_2) \\
& \qquad -8r_1{}^{2m}x_1{}^m r_2{}^{3m} \cos m\phi_2] \\
& +x^3[16x_1{}^{2m}r_1{}^{4m}r_2{}^{2m} \cos m\phi_2 \\
& \qquad -2r_1{}^{4m}x_1{}^{2m}r_2{}^{2m}(1 + 2\cos^2 m\phi_2) \\
& \qquad +8r_1{}^{4m}x_1{}^m r_2{}^{3m} \cos m\phi_2 \\
& \qquad -4r_1{}^{3m}x_1{}^m r_2{}^{4m} \cos m\phi_1 \\
& +x^2[4r_1{}^{4m}x_1{}^{2m}r_2{}^{4m}(1 + 2\cos^2 m\phi_2)^2 \\
& \qquad -32r_1{}^{4m}x_1{}^{2m}r_2{}^{4m} \cos^2 m\phi_2 \\
& \qquad +2r_1{}^{4m}x_1{}^{2m}r_2{}^{4m}] \\
& \quad x^1\,[16r_1{}^{4m}x_1{}^{2m}r_2{}^{6m} \cos m\phi_2 \\
& \qquad -4r_1{}^{4m}x_1{}^{2m}r_2{}^{6m}(1 + 2\cos^2 m\phi_2)] \\
& +x_1{}^{2m}r_1{}^{4m}r_2{}^{8m}
\end{aligned}
$$

In the coefficients of x^6, x^4, x^3, x^2 and x the double-product terms do not necessarily vanish, in comparison with the square terms, as m is increased. These coefficients eventually fluctuate in sign. The coefficients of x^7, x^5 and x^0 are regular (i.e., the double product terms vanish in comparison with the squared terms).

This is the same distribution of regular and irregular coefficients as on the table, which proves that the above assumption

$$r_1 e^{\pm i\phi_1} > x_1 > r_2 e^{\pm i\phi_2}$$

is correct.

From the dominant equation and the table we get from the coefficient of x^5

$$(r_1{}^2)^{32} = 3.4339 \times 10^{30}$$

$$r_1{}^2 = 9.000$$

from the coefficient of x^4

$$(x_1)^{32}(r_1{}^2)^{32} = 1.474 \times 10^{40}$$

therefore $$x_1{}^{32} = \frac{1.474 \times 10^{40}}{3.4339 \times 10^{30}} = 4.292495 \times 10^9$$

$$x_1 = 2.00$$

and from the coefficient of x^0

$$(x_1)^{32}(r_1{}^2)^{32}(r_2{}^4)^{32} = 2.7205 \times 10^{59}$$

$$(r_2{}^4)^{32} = \frac{2.7205 \times 10^{59}}{1.474 \times 10^{40}} = 1.8457 \times 10^{19}$$

$$r_2{}^2 = 2.000$$

The real and imaginary parts of the complex roots are determined from the moduli r_1 and r_2 and the coefficients of the original equation in the following manner:

If $$r_1 e^{\pm i\phi_1} = (u_1 \pm i\, v_1) \quad \text{and} \quad r_2 e^{\pm i\phi_2} = (u_2 \pm i\, v_2)$$

we can multiply out the products of all roots and equate to zero.

$$(x + x_1)(x + u_1 + i\,v_1)(x + u_1 - i\,v_1)(x+u_2+iv_2)^2(x+u_2-iv_2)^2 = 0$$

and get

$$x^7$$

$$+x^6[4u_2 + 2u_1 + x_1]$$

$$+x^5[6u_2{}^2 + 2v_2{}^2 + 8u_1u_2 + u_1{}^2 + v_1{}^2 + 4u_2x_1 + 2u_1x_1]$$

$$+x^4[4u_2^3 + 4u_2v_2^2 + 12u_1u_2^2 + 4u_1v_2^2 + 4u_1^2u_2 + 4u_2v_1^2$$
$$+6u_2^2x_1 + 2v_2^2x_1 + 8u_1u_2x_1 + u_1^2x_1 + v_1^2x_1]$$
$$+x^3[u_2^4 + 2u_2^2v_2^2 + v_2^4 + 8u_1u_2^3 + 8u_1u_2v_2^2 + 6u_1^2u_2^2$$
$$+6v_1^2u_2^2 + 2u_1^2v_2^2 + 2v_1^2v_2^2 + 4u_2^3x_1 + 4u_2v_2^2x_1$$
$$+12u_1u_2^2x_1 + 4u_1v_2^2x_1 + 4u_1^2u_2x_1 + 4u_2v_1^2x_1]$$
$$+x^2[2u_1u_2^4 + 4u_1u_2^2v_2^2 + 2u_1v_2^4 + 4u_1^2u_2^3 + 4v_1^2u_2^3$$
$$+4u_1^2u_2v_2^2 + 4v_1^2u_2v_2^2 + u_2^4x_1 + 2u_2^2v_2^2x_1 + v_2^4x_1$$
$$+8u_1u_2^3x_1 + 8u_1u_2v_2^2x_1 + 6u_1^2u_2^2x_1 + 6v_1^2u_2^2x_1$$
$$+2u_1^2v_2^2x_1 + 2v_1^2v_2^2x_1]$$
$$+x^1\,[u_1^2u_2^4 + u_1^22u_2^2v_2^2 + u_1^2v_2^4 + v_1^2u_2^4 + 2v_1^2u_2^2v_2^2$$
$$+v_1^2v_2^4 + 2u_1u_2^4x_1 + 4u_1u_2^2v_2^2x_1 + 2u_1v_2^4x_1 + 4u_1^2u_2^3x_1$$
$$+4v_1^2u_2^3x_1 + 4u_1^2u_2v_2^2x_1 + 4v_1^2u_2v_2^2x_1]$$
$$+x^0[u_1^2u_2^4x_1 + 2u_1^2u_2^2v_2^2x_1 + u_1^2v_2^4x_1 + v_1^2u_2^4x_1 + 2v_1^2u_2^2v_2^2$$
$$x_1 + v_1^2v_2^4x_1]$$

then from the coefficient of x^6

$$4u_2 + 2u_1 + x_1 = a_1 = 6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and substituting the relations $v_1^2 = r_1^2 - u_1^2$

$$v_2^2 = r_2^2 - u_2^2$$

in the expression for the coefficient of x^5 we obtain:

$$6u_2^2 + 2(r_2^2 - u_2^2) + 8u_1u_2 + u_1^2 + r_1^2 - u_1^2 + 4u_2x_1 + 2u_1x_1 = a_2 = +\,25$$

or

$$4u_2^2 + 2r_2^2 + 8u_1u_2 + r_1^2 + 4u_2x_1 + 2u_1x_1 = 25 \quad . \quad . \quad (2)$$

from (1) $\qquad u_1 = +\,2(1 - u_2)$

substituting u_1 in (2) we get

$$-12u_2^2 + 16u_2 + 2r_2^2 + r_1^2 + 4x_1 = 25$$

and introducing the known values for r_1, r_2 and x_1 we get

$$u_2{}^1 = 1 \qquad \text{and } u_2{}^{11} = \tfrac{1}{3}$$

and

$$u_1{}^1 = 0 \qquad u_1{}^{11} = \tfrac{4}{3}$$

$$v_1{}^1 = \pm 3 \qquad v_1{}^{11} = \pm 2.69$$

$$v_2{}^1 = \pm 1 \qquad v_2{}^{11} = \pm 1.37$$

As there are too many values we introduce them into the expression of the coefficient of x^4

$$4u_2{}^3 + 4u_2v_2{}^2 + 12u_1u_2{}^2 + 4u_1v_2{}^2 + 4u_1{}^2u_2 + 4u_2v_1{}^2 + 6u_2{}^2x_1$$

$$+2v_2{}^2x_1 + 8u_1u_2x_1 + u_1{}^2x_1 + v_1{}^2x_1 = 78$$

and find that only the following values are correct:

$$x_1 = 2 \qquad u_1 = 0 \qquad v_1 = \pm 3$$

$$u_2 = 1 \qquad v_2 = \pm 1$$

The algebraic signs of all the roots are determined by substitution in the original equation.

The roots are $\pm 3i$, 2, $1 \pm i$, $1 \pm i$

PART II. AUTOMOBILES

Chapter 10

STATIC MEASUREMENTS

Before we can study the causes of vibration in an automobile we must know what it is that we are dealing with. This chapter will be devoted to those properties of cars which can be determined by measurement without having to produce vibration in the car. These are the simple factors of

(1) Weight
(2) Center of gravity
(3) Moment of inertia
(4) Spring flexibility
(5) Frame and body stiffness

Weight and Center of Gravity. It is easy to weigh the entire car and also to weigh the wheels and tires separately. In Fig. 68 it is evident, by taking moments about the front wheel, that

$$W_2 \times L = (W_1 + W_2) \times a$$

Therefore, $$a = L \times \frac{W_2}{W_1 + W_2}$$

and similarly $$b = L \times \frac{W_1}{W_1 + W_2}$$

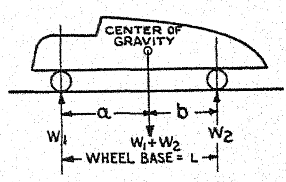


FIG. 68

If the center of gravity of the entire car is required, W_1 and W_2 will be the weights at the front and rear wheels. If we need the center of gravity of the body alone, then

W_1 = weight at front wheels, minus the weight of the wheels, tires and other unsprung parts, and
W_2 has the corresponding value for the rear.

The center of gravity of a car is usually a little to the rear of the center of the wheelbase, so that the rear wheels are a little heavier than the front wheels. This is desirable if we are to obtain the most effective possible braking, which would approach equal braking on all four wheels. Due to the rapid deceleration during braking the car shifts load from the rear to the front wheels, so that, if the wheels are to be equally loaded during braking, the rear wheels should be the more heavily loaded under static conditions.

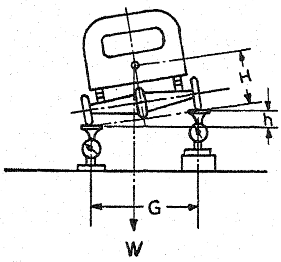


FIG. 69

The height of the center of gravity above the ground may be found approximately by blocking the suspension springs, mounting the car on four scales and then raising the scales on one side as high as is convenient.

Let G = distance between wheels (see Fig. 69)
h = distance one side is jacked up
W = total weight of car

Then when the one side is jacked up an amount h, the center of gravity is displaced sideways a distance $\dfrac{H \times h}{G}$, where H is the height of the center of gravity above the ground. This causes an increase of total load on the two low scales of $\dfrac{W \times H \times h}{G^2}$ so that height of center of gravity (H) = increase of total load on the low side $\times \dfrac{G^2}{Wh}$. For example, if the weight of the car, W = 3500 lb. G = 59 in., and if the total load on the left-hand scales is 1770 lb. before raising the right-hand side and 2040 lb. after jacking up the right-hand side 8 in. we have an increase of 270 lb.

$$\text{Therefore } H = 270 \times \frac{59^2}{3500 \times 8} = 34 \text{ in.}$$

Moment of Inertia. The moment of most interest is that about a horizontal transverse axis through the center of gravity. This is the moment of inertia which is involved in pitching oscillations.

A satisfactory way to measure it is to support the car on a light platform which is in turn supported on knife edges and on springs as shown in Fig. 70.

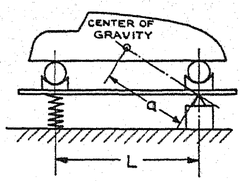


FIG. 70

Then if the car is blocked, so that the springs and tires cannot deflect, the whole assembly can be vibrated and the natural frequency of free vibration can be measured.

If I = moment of inertia of car and platform, lb. in.^{2}
g = acceleration due to gravity = 32.2 ft./sec.^{2}
k = stiffness of the spring in lb./in. deflection
L = distance from pivot to spring, in.
f = natural frequency of oscillation, cycles/sec.

Then $$I = \frac{gkL^2}{4\pi^2 f^2} = 0.82 \frac{kL^2}{f^2} \text{ lb. in.}^2$$

The moment of inertia of the car alone is obtained by measuring the moment of the supporting frame and subtracting it from the total. Finally, if "a" is the distance from the pivot to the center of gravity of the car and W = weight of the car,

Moment of inertia of car about an axis through the C. of G. =
M. of I. about the pivot point $- Wa^2$.

Simpler variations of the test set-up will be obvious, but unless the axis of rotation is definite and free and the spring is accurately calibrated the results will show errors.

Spring Flexibility. Springs may be tested by themselves, but will frequently be stiffer when installed in a car than when they are free. This is due to stiffness in the mounting and is considerable in the case of leaf springs. The stiffness of a leaf spring suspension against roll may be 50% greater than it would be if the springs were supported perfectly. For up-and-down movement an increase of stiffness of 25% due to imperfect support is typical. The best way to measure the spring flexibility is to set the car on four scale platforms and to jack the front and rear bumpers up and down, using jacks fastened to the

floor. The arrangement is shown diagrammatically in Fig. 71. A scale indicates the movement of marks on the fenders and on the wheel hubs, so that spring and tire flexibility both may be measured. By using two jacks, one on each side of the car, the spring flexibility for both up-and-down movement and for roll may be measured. The frame should be jacked about 4 in. above and below its normal position at each wheel. It may be found that the relation between load and

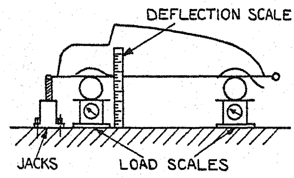


FIG. 71

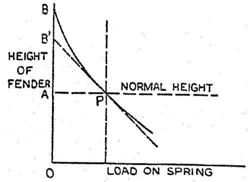


FIG. 72

deflection is not a straight line, but is a curve, such as BP in Fig. 72. The flexibility which is important is then measured at the normal position of the car body by drawing the tangent B^1P which touches the curve at the point of normal height P. The slope of the tangent B^1P gives the normal flexibility in in. per lb. load, or measured the other way, the stiffness in lb. per in. deflection. In many formulae the static deflection of the spring under normal load is used and it is assumed that the spring characteristic is a straight line. In such cases the static deflection to use in the formula must be the "straight line deflection" AB^1 and not the actual deflection AB.

Springs are designed to be a compromise between varying requirements, the most important being that if the springs are too stiff the car will oscillate too quickly and be uncomfortable and also the pressure of the tires on the road will be very variable, which makes steering difficult. Opposed to this are the practical difficulties of finding room for very flexible springs and of providing for big variations in deflection between light and loaded cars if the springs are too flexible.

In some cases it is necessary to provide auxiliary springs which come into action only when the car is heavily loaded. This is shown diagrammatically in Fig. 73. The spring A acts alone when the car is light. Under heavy load the body moves downwards until the stops SS touch the spring B, after which both springs resist.

The characteristic of a spring of this kind is shown in Fig. 74 in which the height of the car body is plotted against the spring load. The point corresponding to the normal position with a light load is at P and the static deflection is BA. If the load is increased, the body drops, picking up the auxiliary spring at Q and reaching the normal fully loaded condition at P^1. For vibrations around the fully loaded position, the spring system has a flexibility given by the slope of QP^1 which would correspond to a static deflection B^1A^1. It will be seen

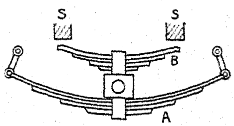


FIG. 73

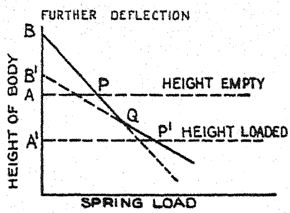


FIG. 74

that by suitable design of the springs we can have the same equivalent static deflection in both the light and loaded positions and therefore, as shown elsewhere, the same frequency of vibration.

Frame and Body Stiffness. The important case in which deflections of the car body are concerned is that of twisting of the body about a longitudinal axis. This twisting is set up by wheel shimmy and was at one time a common occurrence at high speed, showing up particularly as vibration of headlights and front fenders as well as wheel shimmy itself. The flexibility of a car body in twisting is most difficult to calculate and in practice it can be found only by test and experience.

The preferable method of test is to support the body at the four points of attachment of the springs. Where a leaf spring is attached at two points, the forward point is chosen, since this includes the part of the frame chiefly involved in twisting oscillations and avoids considering the extreme rear end, which may be very flexible but does not usually oscillate in this way. One front corner is then raised or lowered 1 in. and the force required to do this is measured. This force is conveniently referred to as the rigidity in lb. per in.

Wheel Alignment. It is convenient to include here certain properties of the car which are obtainable from the design or by measurement. These quantities were defined by the Research Sub-Committee

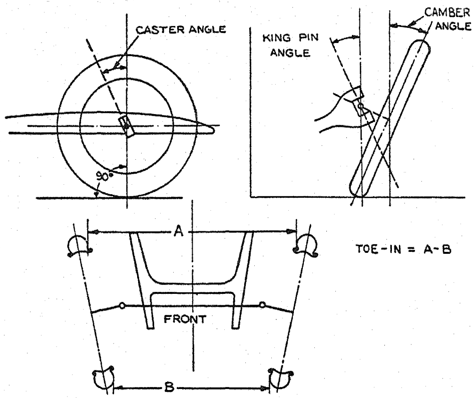


Fig. 75

of the S.A.E. in 1935. They are caster angle, kingpin angle, camber angle and toe-in, and are illustrated in Fig. 75. Typical values are:

Caster angle	2°
Kingpin angle	5°
Camber angle	0°
Toe-in	$\frac{1}{16}$ in.

In practice they vary considerably.

Chapter 11

DYNAMIC MEASUREMENTS

Dynamic measurements which are of interest in studying car vibrations are of two major kinds:

(*a*) Road and proving ground tests
(*b*) Laboratory tests

It is evident that road tests are the final ones which demonstrate whether or not the car meets the actual driving conditions for which it is intended. The vast majority of testing is of this kind. It is also easy to see that road tests are the most difficult to correlate scientifically because:

(*a*) The primary interest is in the effect on passengers or freight, which is somewhat of an intangible as it cannot be directly measured.

(*b*) The standards considered satisfactory change from year to year and the instruments which are devised to "measure comfort" also vary both in detail and in general principles. Since comfort itself is not measurable, the instruments measure various combinations of acceleration, frequency, motion, etc., which appear to be closely correlated with the average impressions of many observers. Ideas on this matter change rapidly.

(*c*) The conditions of road tests vary widely and usually are not known. Sometimes a fixed track is used for comparative purposes, but its surface will be arbitrary and not suitable for a simple scientific description. The proper kind of road tests to give results of value also changes with the years as the general quality of roads improve.

Therefore, lengthy discussion of road testing, important as it is, will be omitted here and reference made only to general results obtained where necessary in connection with laboratory tests. The

many road testing instruments and the developments of that art are frequently and fully described in automotive periodicals, to which the reader is referred.

Laboratory tests of the dynamics of cars are also varied, but there are two kinds of tests which are of fundamental interest. The first test concerns the effect of a single bump on the car. The second concerns the effect of a regularly repeated bump. In the first case the chief object of the test is to set up and study the natural oscillation of the car as it steadies itself after a sharp bump. In the second case the object is to set up a regular forced oscillation and to study the behavior of the car under this condition.

The actual condition of running on a road is that there is a continuous but varying forced oscillation, due to the irregular surface and at the same time the natural free oscillation of the car is continually being set up and dying out as the forced oscillation changes.

Free Oscillations. The simple basic test consists of raising one wheel of the car, usually about 4 in., letting it drop and recording the motion of the body and of the wheel. The wheel may be raised on a jack which can be tripped, or it may be held up by a wire, which is cut. In other cases, the wheel may be held at its normal height over a shallow pit and then allowed to drop in. In any case the effect is that of a sudden change in road level under one wheel. It would be equally interesting to raise one wheel suddenly, but it is so much easier to drop it that the test is generally made that way.

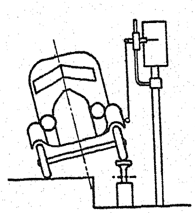


FIG. 76

Fig. 76 shows one method of making the test. One wheel is raised on a jack. The position of the body is recorded on a revolving cylinder by a pencil attached to one of the fenders or some other convenient point. By attaching the pencil to the wheel hub a record is also obtained of the wheel motion when the jack is tripped.

The type of record obtained is shown in Fig. 77. It will be noted that both body and axle perform damped oscillations but with very different frequencies. The frequency of the body will normally lie between 1 and $2\frac{1}{2}$ cycles per sec. The frequency of the axle will be nearer to 10 cycles per sec.

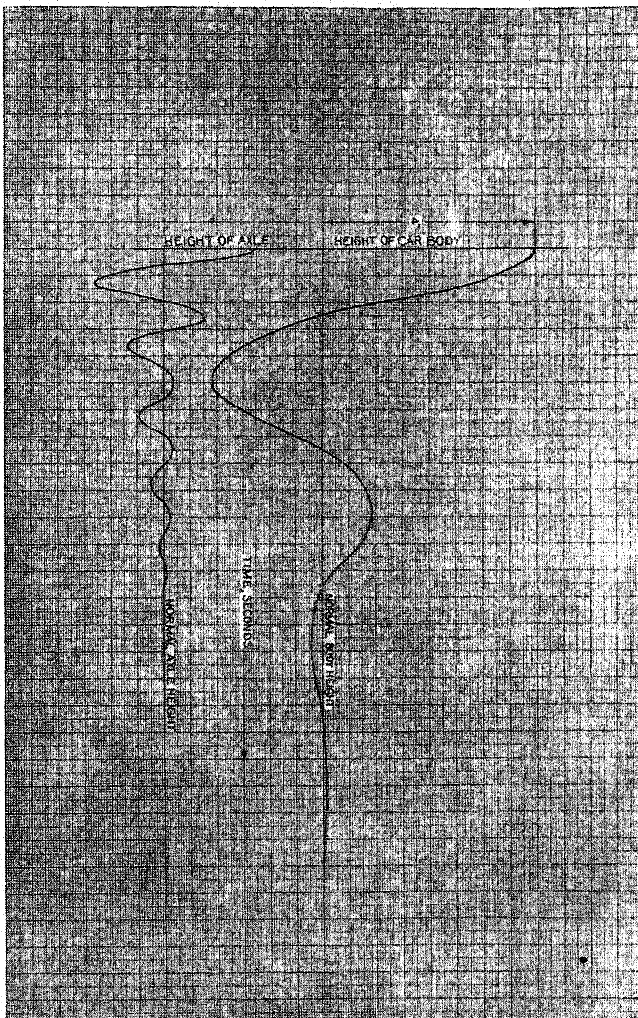


FIG. 77. AUTOMOBILE STATIC MEASUREMENTS

Considerable study may be applied to the diagram. The springs (and shock absorbers) should make the body return to its normal position with a frequency low enough to avoid discomfort to passengers with normal variations in surface. With a normal total amplitude of about 4 in. to be provided for, it is found that the frequency should be not over 1.33 per sec. (80 per min.).

Representative values for 1936 were 75 per min. for the front springs and 69 per min. for the rear springs.

Also the rebound should be as small as possible so that the effects of successive bumps will have as little chance as possible to add to each other.

Then the shape of the body curve should be studied to see if it shows unduly sharp curvature at any point. At the maximum spring compression it is particularly likely to bend very sharply due to some binding or improper action of the springs. This means a high momentary acceleration of the body which is likely to cause discomfort and should lead to careful examination of the springs.

The axle curve should also be reasonably damped, for two reasons. First the vibration should not allow the axle to rise more than about $\frac{1}{2}$ in. above the normal position during rebound. If it does, the pressure on the road becomes insufficient for proper steering. If the tire leaves the ground, steering is of course impossible. The second reason for damping is to avoid large axle vibration when the body is at its lowest point and therefore most affected by spring compression. Usually the axle vibration is still appreciable when the body reaches its lowest point. Then it is desirable to avoid the axle being also at its lowest point. If it is, the body can drop so much farther, resulting in higher accelerations and more rebound. In Fig. 77 it will be seen that the axle is up when the body is at its lowest point. This is desirable because the body movement is clearly reduced and also because the spring compression is a maximum at a time when the axle tends to jump and the tire is held down for at least a whole cycle of the axle movement, thereby tending to stop any skid which was started by previous rebounds. A very small change in the spring will change the relation between the peaks of body and axle motion and may appreciably improve the behavior of the car.

If body motion curves of this kind are taken for both front and

rear wheels, interesting studies can be made of the action of a car in which the front and rear wheels pass successively over a bump.

First, let us suppose that the front and rear spring systems are independent of each other. This is true if the inertia of the car has the proper relation to the wheelbase.

If the car has, say, a 10-ft. wheelbase and is moving at 60 m.p.h., the rear wheels will hit a bump $\frac{10}{88} = \frac{1}{8.8}$ sec. after the front wheels. Now let us plot the body motions at front and rear on the same picture, with the interval of $\frac{1}{8.8}$ sec. between their starts. In Fig. 78 this has been done for three cases: (*a*) equal natural frequencies for front and rear; (*b*) lower frequency (softer springs) in rear; (*c*) lower frequency (softer springs) in front.

It will be seen that the vertical distance between the curves at any time, such as *PP*, represents the difference in height between the front and the rear ends of the body; that is, it is a measure of the pitching of the car. Pitching tends to force the passengers' heads back and forth and gives the impression of an uncomfortable ride.

Starting with Fig. 78 (*a*) in which the springs have equal flexibilities, front and rear, it is clear from (*b*) that if the rear springs are softened, the pitching will get worse. In (*c*) the rear springs are a little stiffer and it will be seen that the pitching is greatly reduced. This approaches quite closely to a simple up-and-down oscillation of the car, the "boulevard ride." The effect of different weight distributions, shock absorbers, etc., can be investigated by similar methods, but as they are more complicated the theory of their action will be considered in a later section.

Forced Oscillations. These are important when a regularly repeated disturbance has a frequency near enough to one of the natural frequencies of the car to build up oscillations of considerable size. Two interesting examples are the effect of washboard roads at slow speeds and of the long waves in concrete roads at high speeds. A washboard road generally excites a vibration of the dead weight of the wheels, which are held between the tires and the suspension springs. These may have a natural frequency of the order of 400 to 600 per min. If the motion becomes violent, the tires will leave the ground

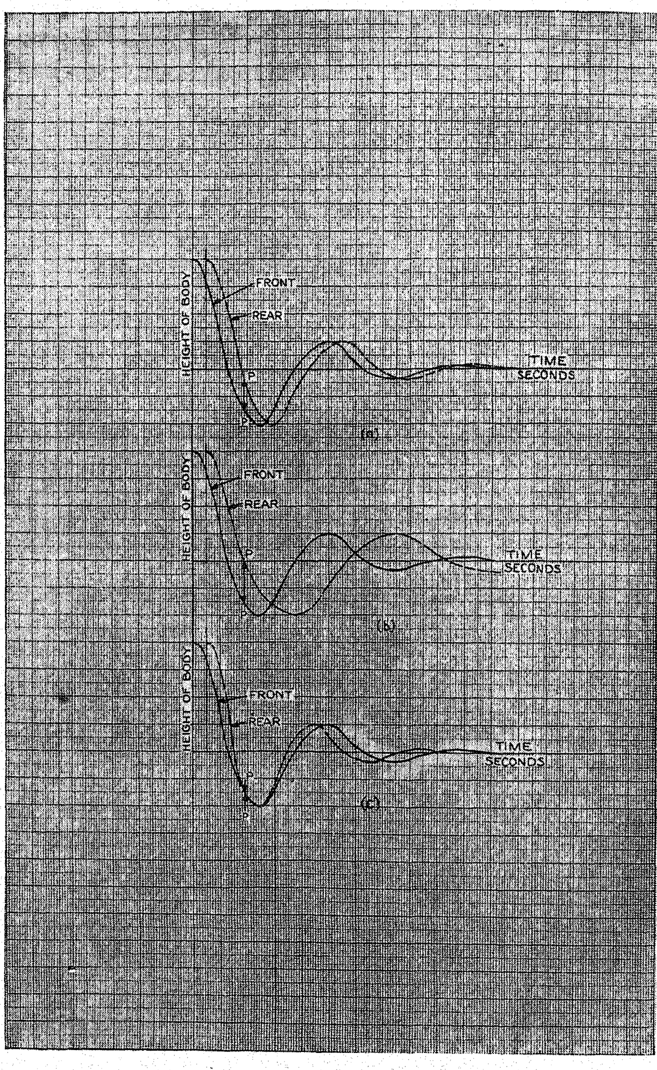


FIG. 78. AUTOMOBILE STATIC MEASUREMENTS

and the frequency will become less, so that a reduction in speed may not stop the oscillation immediately.

The long heat waves in concrete highways may excite a motion of the whole car on the tires, with a frequency of 250–350 per min., or a motion of the suspension springs with frequencies of the order of 60–150 per min.

To study these in the laboratory the car wheels may be mounted on pivoted rollers which are regularly raised and lowered at proper intervals by motor-driven cams. The car may then be driven at any speed on the rollers and the oscillations recorded by pencils on revolving drums.

Shock Absorbers. The value of some kind of shock absorbers on the suspension of a car is easily seen. They keep oscillations from persisting and possibly adding to subsequent ones, reduce the rebound from shocks, and limit oscillations which are built up by resonance between some regular disturbance and a natural frequency of the car. In the early days most cars had leaf-spring suspensions with considerable friction in the springs and this friction was relied upon to perform the functions now left chiefly to shock absorbers. The friction of leaf springs is very variable since they may be dry and rusty or well oiled. There is also the tendency to stick entirely and give a jerky motion. Another disadvantage of friction is that it reverses abruptly whenever the direction of motion changes. Therefore it produces large changes of force and hence large changes of acceleration at the end of each swing. These changes are unpleasant and give the impression of a rough ride.

Refinements of design and construction have produced leaf springs which, if properly greased, may be reasonably satisfactory but it has become general to prefer separate shock absorbers designed purely for this purpose.

These absorbers are commonly arranged to obtain oil damping either by rotating vanes in oil or by pumping oil through small passages. In both cases a problem has been the change of viscosity of oil with change in temperature. On hot days the shock absorbers would not be adequate, while on very cold days they would be almost solid. The designs in which oil is pumped through small passages have the advantage from this point of view because they can be adjusted to use

very light oil and the viscosity of light oil changes much less than the viscosity of heavy oil with wide changes of temperature.

The characteristic generally found desirable in this kind of device is one which increases the restraint with speed up to a certain value and then provides a more nearly constant restraint as the speed of action increases.

Fig. 79 shows a typical curve, but there is no one best characteristic since different cars require different treatment.

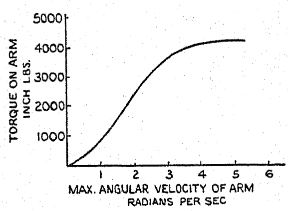


FIG. 79

It is easy to see that a simple shock absorber, in which the force increases roughly with the rate of rotation of the lever, has disadvantages so far as a comfortable ride is concerned.

The aim in any suspension is to have the wheels follow the surface of the road as closely as possible and to have the car body ride on a uniform level path, with as little disturbance as possible from the irregularities of the road. To approach this result it is of course necessary first of all to have soft springs. Then, if the springs are damped by the addition of shock absorbers, the absorber force will act to make the body follow the fluctuations of the wheels more closely and will therefore give a rougher ride on a rough road. The general principle of damped simple harmonic motion applies: whenever the springs improve the ride, damping makes it worse. The only value of the damping is to reduce the car motion at times when the spring action would increase it, due to resonance, rebound and similar effects. It is worth following this out in more detail.

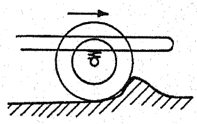


FIG. 80

In Fig. 80 a wheel meets a bump in the road. A flexible spring system will allow the wheel to pass over the bump with the minimum of disturbance to the car body. If the spring is damped, the damping force will add to the spring force in accelerating the body as the wheel rises. After the wheel passes the top of the bump, the damping force retards the drop of the wheel. After this the damping acts to stop the body continuing to oscillate and is beneficial.

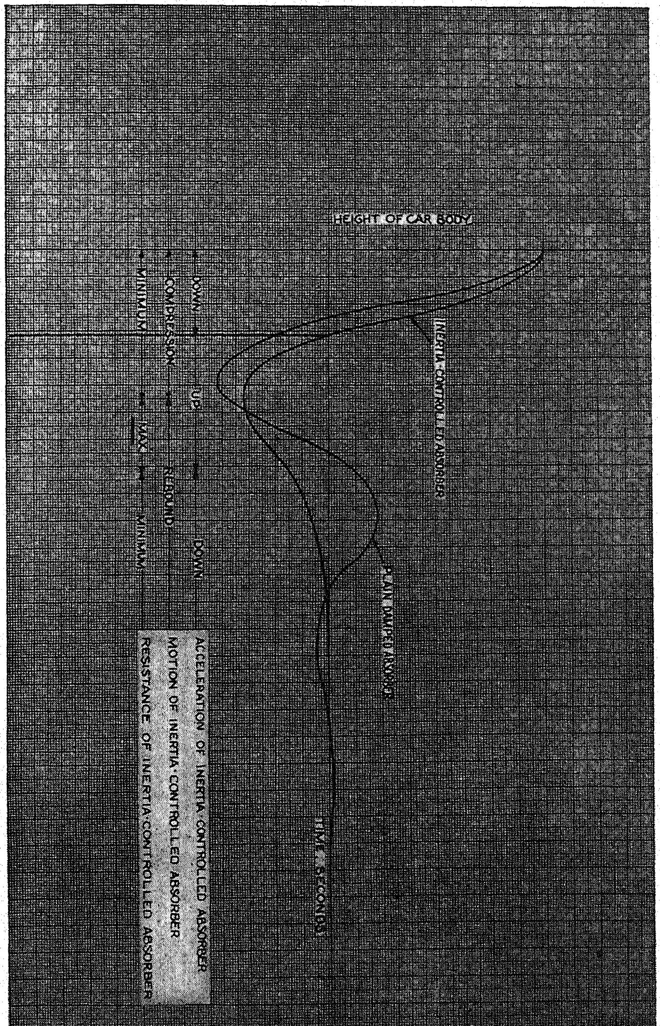

HEIGHT OF CAR BODY
INERTIA CONTROLLED ABSORBER
DOWN
UP
DOWN
COMPRESSION
REBOUND
MINIMUM
MAX
MINIMUM
ACCELERATION OF INERTIA CONTROLLED ABSORBER
MOTION OF INERTIA-CONTROLLED ABSORBER
RESISTANCE OF INERTIA-CONTROLLED ABSORBER

FIG. 81

It will be seen that as long as the body rides steadily, the shock absorbers should not act; hence absorbers have been built which damp the spring action only when the body has been accelerated upwards. This is done by designing the shock absorber with a spring-controlled weight, movement of which restricts the flow of oil in the rebound side of the absorber. If the absorber, which is fastened to the car body, is steady, there is minimum resistance to spring motion. If the absorber is accelerated, considerable resistance is introduced to the rebound; that is, to the separation of the wheel and body.

Following the action of this "inertia-controlled" absorber as a wheel passes over a bump, we see that the wheel rises with minimum resistance and minimum force is transmitted to the body. The upward acceleration causes the absorber to retard the drop of the wheel as in the case of plain damping.

The inertia-controlled shock absorber shows to the best advantage when a wheel drops or passes over a low spot. To take the simple case of a wheel dropping to a lower level; the wheel drops with minimum resistance and hence reaches its new level as quickly as possible. This means that the body loses its spring support for the shortest possible time and drops comparatively slowly. As it drops, it is gradually stopped by the upward force of the spring, which produces an upward acceleration, thereby setting the absorber for maximum restraint against rebound before the rebound takes place.

In Fig. 81 drop curves for an automobile body are sketched to show the difference in action of inertia-controlled and plain absorbers. The superiority of the inertia control in this case is very clear.

Shock absorbers are frequently built with pressure relief valves to limit the damping force to a predetermined value. This makes the absorber much less dependent on the viscosity of the oil and therefore on temperature. It has also the advantage that rapid oscillations of the wheel between the tire and suspension springs do not transmit such large damping forces to the car body.

Chapter 12

SPRING SUSPENSION AND WEIGHT DISTRIBUTION

In the preceding chapters it has been shown that a comfortable level ride of a car body requires (*a*) soft springs, to give a low frequency; (*b*) nearly equal frequencies of front and rear springs, to avoid pitching; and (*c*) frequency of front springs slightly lower than that of rear springs to compensate to some extent for the front wheels hitting a bump slightly before the rear wheels.

The theory of oscillations on the springs will now be reviewed. The basis of the theory has been given in the general study of systems of two degrees of freedom. It was shown that a car body supported on front and rear springs oscillates in a combination of two "normal" ways. Each normal way is the same as if the body were pivoted at a certain fixed point. These "pivot points" and the corresponding frequencies are given by equations (8) and (9), Chapter 4.

If the front and rear springs have equal static deflections, the pivot points are at the center of gravity and at infinity; that is, the oscillations are a combination of pitching around the center of gravity and vertical bouncing. If the front springs are stiffer one pivot is in front of the car and the other is a little behind the center of gravity. It is common to refer to pitch and bounce in this case also. We mean by pitch an oscillation about the pivot a little behind the center of gravity. Similarly, bounce is the oscillation about the pivot which is some distance ahead of the car.

An important ratio is $\frac{k^2}{ab}$; that is, the ratio of the radius of gyration, squared, to the product of the distances from the center of gravity to the front and rear wheels.

With equal spring deflections;

$$\frac{\text{Bounce frequency}}{\text{Pitch frequency}} = \sqrt{\frac{k^2}{ab}}.$$

This is seen from equations in Chapter 4 and since

$$\frac{\text{Bounce frequency}}{\text{Pitch frequency}} = \sqrt{\frac{\frac{g}{w}(s_a + s_b)}{\frac{g}{w}\frac{s_a a^2 + s_b b^2}{k^2}}}$$

and $as_a = bs_b$ or $s_a = \frac{b}{a} s_b$

$$\text{Therefore} \quad \frac{\text{Bounce frequency}}{\text{Pitch frequency}} = \sqrt{\frac{s_b\left(\frac{b}{a} + 1\right)}{s_b\left(\frac{ab + b^2}{k^2}\right)}} = \sqrt{\frac{k^2}{ab}}$$

Hence an increase in $\frac{k^2}{ab}$ will decrease the pitch frequency and result in a smoother ride up to the point where $\frac{k^2}{ab} = 1$ and the frequencies are equal. This is the condition where the front and rear spring motions are independent of each other. The condition $\frac{k^2}{ab} = 1$ is approached more or less nearly in most cars which have good riding characteristics. Opinions differ as to how closely the relationship need be approached. Values of $\frac{k^2}{ab} = 0.75$ have been used in good riding cars.

It is interesting to note that $\frac{k^2}{ab}$ can be increased in two ways; either by increasing k^2 which means putting weight near the ends of the car, or by decreasing ab. This may be done by decreasing the wheel base, or by arranging the center of gravity farther from the center of the wheel base. For example, if the center of gravity is moved from the middle of the wheelbase to a point $\frac{2}{3}$ of the way from front to rear, with no change in wheelbase or radius of gyration $\frac{k^2}{ab}$ will be increased over 11%. This is to be considered in comparing rear engine with front engine cars, since the weight distribution may

be quite different. The advantage of a value of $\frac{k^2}{ab}$ near to unity arises in considering steering as well as vertical oscillations.

If the car is taking a curve with the wheels near the skidding point, a bump or a slight change in steering at the front wheels might affect the rear wheels enough to make them skid. Also a disturbance at the rear might badly affect front wheel steering. However, if $\frac{k^2}{ab}$ is approximately unity the changes of one wheel have no effect on the other and the safety on curves at high speed is increased accordingly.

Chapter 13

SHIMMY

In its worst form shimmy is a vibration, chiefly of the front end of a car, in which the frame and front fenders vibrate torsionally and at the same time the wheels vibrate about the steering-knuckle pivots. It is caused chiefly by unbalance of the front wheels or by an irregular road surface which excites torsional vibrations in the frame. Once set up, shimmy usually tends to sustain itself and can be stopped only by reducing speed.

The two most noticeable components of shimmy are tramp and wheel wabble. Tramp is a vibration of the front axle assembly, in a

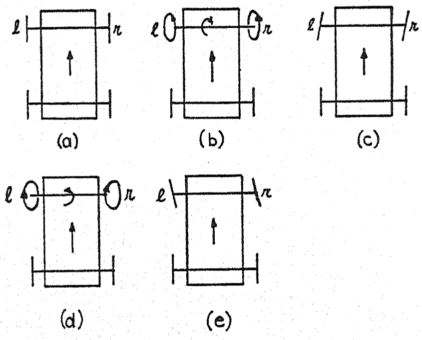


FIG. 82

vertical plane, about a point usually somewhat above the center of the axle. Wheel wabble is an angular vibration of the wheels about the steering-knuckle pivots.

At high speed the gyroscopic forces due to rotation of the wheels are sufficient to keep the tramp and wheel wabble in step with each other and to produce a self-sustaining oscillation.

To illustrate the reason for this, let us consider the plan view of a car traveling at high speed. In Fig. 82 (*a*) shows the normal condition of the car, with the left and right front wheels denoted by " l " and " r " respectively. In (*b*) the front axle is shown tilted, with the

left wheel raised. As the arrows indicate, the rotation of the wheels gives us a counterclockwise angular momentum which tends to set up an equal and opposite clockwise momentum in the car structure. If the front wheels have some degree of looseness in the steering mechanism, they will turn about the steering-knuckle pivots into the position shown in (*c*). Now the right wheel rises and the wheels turn back to proper alignment as in (*d*). Here the rotation of the wheels about the axle gives the clockwise angular momentum and the wabble gives a counterclockwise angular momentum. The axle returns to level as the wheels take their full wabble to the left as shown in (*e*) and the cycle then repeats.

The motion may be started by unbalance in the front wheels. This produces a periodic tilting of the front axle or tramp and the gyroscopic action of the wheels causes them to wabble as described above. This kind of shimmy will start as soon as the speed is sufficient to allow the unbalanced centrifugal forces and gyroscopic forces to overcome the frictional restraints and will then become progressively worse with speed. The shimmy will be accentuated when the frequency of rotation of the wheels approaches the natural frequency of torsional oscillation of the car body. This is because as the front wheels vibrate from side to side they twist the frame and produce a forced torsional vibration in it.

The shimmy may be started also by a road irregularity which raises one of the front wheels. The gyroscopic effect starts a wabble and the resulting sinuous motion produces torsional vibrations of the body. These torsional vibrations sustain the tramp and hence the shimmy if the speed is high enough to produce sufficient gyroscopic forces. The source of energy for this sustained vibration is of course the motive power of the car acting through the wabble of the wheels to shake the front end of the car from side to side. The frequency of this sustained shimmy is nearly equal to the natural period of torsional vibration of the body.

Considerable work has been done to study torsional body vibrations. It has been found that the nodal point of no vibration is usually near the middle of the car or around the front seat. The front end is also quite flexible compared with the rear end.

It was thought at one time that the vibration was confined to the

front end or that it was in the frame alone, but tests have shown that the entire car body is involved. Tests made with specially rigid frames show that this change alone does not cure front-end vibration and that the more rigid front end of the frame may produce a harsher ride. In order to obtain the required stiffness the body must be very firmly attached to the frame and must be designed with torsional rigidity in mind. In early cars the engine was rigidly fastened to the frame and this not only gave rigidity to the flexible front-end members, but also provided mass which tended to reduce the liability to shake. When engines were flexibly mounted, both these advantages were lost. In order to increase the torsional moment of inertia of the light front ends of cars a great variety of designs has been used. Spare wheels, batteries and toolboxes have been carried in the front fenders, radiators have been attached to the frames in order to use their mass, and front bumpers have been weighted.

An interesting design, used particularly on heavy cars, consists of spring-supported weights attached to the ends of the front bumper. By proper choice of spring deflection the weights can be timed to about the frequency of the body. Then if this frequency is excited, the vibration takes place almost entirely in the spring-supported weights and the body does not vibrate appreciably. When these devices were developed it was found that the frequency required was from 500 to 600 cycles per min. and that weights of approximately 11 lb. were sufficient for a 5400-lb. car. The weights are damped with oil which also keeps the parts lubricated.

Trouble with shimmy has been reduced markedly with greater spring deflections, better wheel balance, softer tires and better body design as well as improved steering mechanisms, so that it is now rather uncommon.

In order to get some idea of the magnitude of the gyroscopic forces which enter into shimmy let us consider a typical light car wheel which with tire and brake drum weighs 45 lb. and has a radius of gyration of 10 in. The moment of inertia is

$$I = 45 \times \frac{(10)^2}{(12)} = 31.25 \text{ lb. ft.}^2$$

and

$$\frac{I}{g} = \frac{31.25}{32.2} = 0.97.$$

At 50 m.p.h. with a 28-in. diameter wheel, the angular velocity is

$$\omega = 50 \times \frac{88}{60} \times \frac{12}{14} = 63 \text{ radians/sec.}$$

If the wheel is tilted with an angular velocity Ω, the gyroscopic couple required is

$$\frac{I}{g} \times \omega \times \Omega = 0.97 \times 63 \times \Omega = 61\Omega$$

Suppose the wheel hits a bump which causes it to bounce 4 in. and that the average spring load, taken between the normal position and a 4-in. rise, is 1200 lb. The initial angular velocity caused by the bump is roughly $\Omega = 5$ radians per sec. in order to give 4 in. motion. Thus the maximum gyroscopic couple is

$$61\Omega = 61 \times 5 = 305 \text{ lb. ft.}$$

Since both wheels turn the same amount, the total couple is 610 lb. ft., which tends to turn the car to the right if the front left wheel rises.

Chapter 14

ENGINE MOUNTING

It has not been found possible to build internal combustion engines which are completely free from all vibration. This vibration is transmitted to the frame of the car through the engine mounting and drive. The object of a good mounting is to reduce the amount of vibration transmitted so that it will not be objectionable. It has been found that the correct use of rubber mountings not only eliminates the annoyance of vibration to the passengers, but at the same time lengthens the life of the chassis and body considerably by largely removing the destructive effects of vibration.

The problem will be illustrated with reference to a 4-cylinder 4-cycle automobile engine. The vibration is due to a secondary piston inertia force with a frequency twice that of the crankshaft revolutions. This arises because a piston on a crankshaft does not move in an exactly simple harmonic motion. A pair of pistons may balance so far as primary motions are concerned; that is, one will be up when the other is down, but the center of gravity of the pair fluctuates slightly as they move. This fluctuation approaches very closely a harmonic motion of twice engine frequency. The center of gravity of the whole engine will remain stationary unless some outside force acts on the engine; therefore, if the pistons oscillate with double frequency the rest of the engine will tend to oscillate with the same frequency but in the opposite direction.

The unbalanced force, due to an unbalanced weight W' reciprocating with a semi-amplitude R with an angular velocity ω (which is twice the angular velocity of the crankshaft) has a value

$$\frac{W'}{g} R\omega^2.$$

If the engine mounting has a natural frequency corresponding to an angular velocity ω_0, then it has been shown that the force transmitted to the supporting car frame is

$$\frac{1}{1 - \dfrac{\omega^2}{\omega_0{}^2}}$$

of the disturbing force.

In this case, therefore, the force transmitted to the frame will be

$$\frac{W'}{g} R \frac{\omega^2}{1 - \dfrac{\omega^2}{\omega_0{}^2}}$$

But $$\omega_0{}^2 = \frac{kg}{W}$$

where k = stiffness of the engine mounting
W = weight of total engine
g = acceleration due to gravity.

Therefore the transmitted force is

$$\frac{W'}{g} R \frac{\omega^2}{1 - \dfrac{\omega^2 W}{kg}} = \frac{kW'R\omega^2}{kg - W\omega^2}$$

If the engine were rigidly mounted, without springs, the transmitted force would be the whole disturbing force of the pistons, or

$$\frac{W'}{g} R\omega^2$$

The force is reduced by the springs whenever the angular velocity of the disturbing force is more than 1.41 times the angular velocity for resonance of the support; that is, the springs help for all speeds more than 41% above the engine speed at which there is resonance in the support. In applying this principle, it is clear that the natural period of the engine on its support should be at a point well below the driving range of the car. Also, as has been shown above, damping

merely increases the transmission of force in the range in which a spring mounting is effective. Therefore, the engine mounting should contain as little damping as is found practicable, the only object of damping being to reduce momentary oscillations during starting and stopping and those produced by rough roads.

In the practical application of this theory it is necessary to recognize that the unbalanced force does not always come just at the center of gravity of the engine and that the fore-and-aft springs on which the engine is mounted may not deflect equally. In this case the engine will vibrate angularly as well as up and down and in general there will be two natural periods at both of which resonance can occur. A simple practical method used to design mounting springs is as follows:

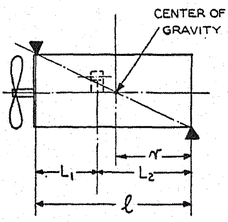


FIG. 83

(1) Assume that the engine pivots at the rear mounting spring. Design the front spring to give a suitable natural period of vibration.

(2) Assume that the engine pivots at the front spring. Design the rear spring to give a suitable natural period, generally the same as for the front spring.

Let r = distance from rear support to center of gravity (Fig. 83)
l = distance between supports
k = radius of gyration of engine about a transverse axis through its center of gravity
k_1 = stiffness of the front engine mounting spring, which it is required to design
W = weight of engine
g = acceleration due to gravity

Then, if the rear spring is taken as a pivot, torque per unit angular deflection $= k_1 l^2$.

Moment of inertia about an axis through rear support

$$= \frac{W}{g}(r^2 + k^2)$$

$$\omega^2 = \frac{\text{torque per unit deflection}}{\text{moment of inertia}} = \frac{k_1 l^2 g}{W(r^2 + k^2)}$$

$$\text{Natural frequency} = f_1 = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_1 l^2 g}{W(r^2 + k^2)}}$$

Hence, if f_1 is a chosen frequency well below the operating range, the required stiffness of the front spring is

$$k_1 = \frac{4\pi^2 f_1{}^2 W(r^2 + k^2)}{l^2 g}$$

The rear spring is calculated in an exactly similar way by putting r = distance from *front* spring to center of gravity.

Example. In an actual engine

r = distance from rear support to center of gravity = 1.94 ft.
l = distance between supports = 3.50 ft.
k = radius of gyration of engine about a transverse axis through the center of gravity = 1.20 ft.
W = weight of engine = 590 lb.

Suppose we choose an engine speed of 150 r.p.m., which is well below the driving range of the car, as suitable for resonance of the engine mounting. The unbalance is a secondary inertia force with a frequency twice engine speed, or 300 per min. or 5 per sec. Then the stiffness of the front mounting will be

$$k_1 = \frac{4 \times 3.14^2 \times 5^2 \times 590(1.94^2 + 1.20^2)}{3.50^2 \times 32.2}$$

$$= 7682 \text{ lb. per ft.} = 640 \text{ lb. per in. deflection.}$$

A similar calculation for the rear support gives a required stiffness of 477 lb. per in.

In the case of this engine the secondary piston inertia forces are equivalent to a weight of 0.6 lb. reciprocating with a stroke of $4\frac{3}{4}$ in. at twice crankshaft speed and at a distance 2.33 ft. ahead of the rear engine mounting. It is, therefore, 0.39 in. ahead of the center of gravity.

At an engine speed of, for example, 500 r.p.m., the secondary frequency is 1000 r.p.m. and the inertia force is

$$\begin{aligned} &\frac{W'}{g} R\omega^2 \\ &= \frac{0.6}{32.2} \times \frac{4.75}{2 \times 12} \times \left(2\pi \times \frac{1000}{60}\right)^2 \\ &= 40 \text{ lb.} \end{aligned}$$

Taking moments about the rear support, the front support would, if rigid, be subjected to an oscillating force of

$$40 \times \frac{2.33}{3.50} = 27 \text{ lb.}$$

The spring suspension, designed for resonance at 150 r.p.m. gives at 500 r.p.m. a reduction of this force in the ratio

$$\frac{1}{1 - \dfrac{\omega^2}{\omega_0{}^2}} = \frac{1}{1 - \left(\dfrac{500}{150}\right)^2} = \frac{1}{1 - 11.1} = -0.099$$

The force transmitted to the car body at the front support is, therefore, reduced by the mounting to $0.099 \times 27 = 2.7$ lb. which cannot be felt in the car.

Torque Reaction. When the engine produces a torque on the drive shaft the shaft reacts with an equal and opposite torque on the engine. It is, therefore, necessary to hold the engine from turning due to this "torque reaction." The torque is irregular and, therefore, if the engine is held rigidly against turning, these irregularities will be transmitted to the car body. This may be overcome by using a flexible support, just as in the case of the unbalanced forces.

For torsional oscillations the frequency is

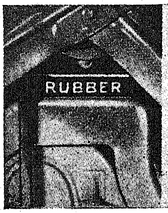
$$f = \frac{1}{2\pi}\sqrt{\frac{Tg}{I}}$$

where T = torsional resistance per radian displacement in lb.-ft. ($=57.3 \times$ resistance per degree of twist)

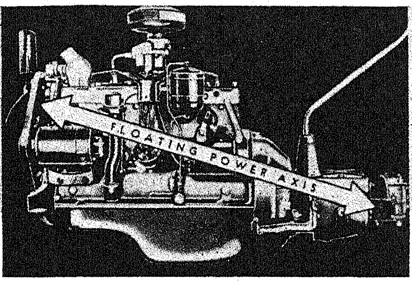
I = moment of inertia of the engine about the axis of oscillation (in lb.-ft.^{2})

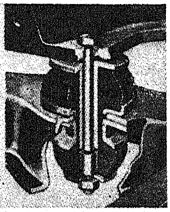
$g = 32.2$

Taking a suitable speed for resonance as 200 r.p.m. which is well below the normal driving speed, the resonance frequency will be 400 per min. (since there are two peaks of torque per revolution in a 4-cylinder engine) or 6.66 per sec.

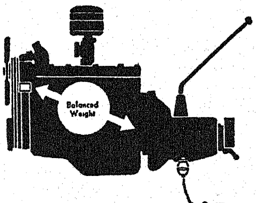


At the front, the engine is suspended by a mounting of live rubber bonded to steel.





Low at the rear of the engine there are these spool-shaped mountings of live rubber.



Two mounting members, one on each side of the transmission.

FIG. 84

The moment of inertia of the engine is found experimentally to be 445 lb.-ft.^{2}

$$\text{Therefore } T = \frac{4\pi^2}{g} f^2 I$$
$$= 1.226 \times (6.66)^2 \times 445$$
$$= 24{,}200 \text{ lb.-ft./radian}$$
$$= 422 \text{ lb.-ft. per degree}$$

This was readily obtained by means of a flat leaf spring.

There is one more very important phase of the problem of taking care of torque reaction in the engine mounting. If the rocking oscillation of the engine displaces the center of gravity, this will result in a reaction on the frame due to the center of gravity of the engine oscillating from side to side. If we fix the axis of rotation of the engine so that it passes through the center of gravity, this reaction is eliminated.

Referring again to the 4-cylinder automobile engine it will be seen from Fig. 83 that the front and rear mountings fix a torsional axis which passes through the center of gravity. The inclination of this axis is fixed so as to minimize the movement of the universal joint. Some practical details are illustrated in Fig. 84, which shows Chrysler Floating Power.

PART III. STREET CARS

Chapter 15

STREET CAR VIBRATIONS

Development of the PCC Car. Developments in recent years on street car design center around the work of the Electric Railway Presidents' Conference Committee which was formed to investigate improvements in the design of street cars. The research work of this committee and the coordination of the results in commercial cars has been carried on by the Transit Research Corporation. At the same time, improved designs have been proposed and built by independent builders and operators.

Since one of the major objectives of this recent development was the reduction of noise and vibration in the conventional street car, some of this work will be considered as an example of what can be done by coordinated development. It will, of course, be understood that it is possible here to summarize only rather briefly the developments which refer to noise and vibration, and that much other development work has been done which will not even be mentioned.

The program of research in the reduction of noise and vibration has been divided generally into six parts.

(1) The observation of cars in service and the development of instruments to measure their vibrations so that accurate comparisons could be made which will be the basis for improvement.

(2) Tests on the effect of vibration on passenger comfort to determine what combination of vibrations in regard to frequency, amplitude and direction were tolerable from the passenger's point of view. This work was done principally on a vibrating table under laboratory conditions.

(3) Observation of a small model car under laboratory conditions. This part of the work was interesting and gave those who worked on it

much insight into the vibration problems involved, but the vibrations tended to be so complex that the results are difficult to analyze and were of less value than had been anticipated.

(4) Development of wheels on a wheel-testing machine. It was realized that solid steel wheels in the older type of street cars were responsible for much of the noise and high frequency vibration which appear in the car body. A wheel-testing machine, described below, was developed for comparative tests on various designs of flexible wheel.

(5) Construction and test of new truck designs. After considerable preliminary research, principles were arrived at which it was believed should be followed in the design of improved trucks. As in most other particular cases, the application of these principles presented numerous problems so that it was felt necessary to test experimental trucks before any were built for commercial service. These tests indicated changes and improvements which were incorporated in the first commercial product.

(6) After the first development, the principal work of the Transit Research Corporation was the observation of trucks in service and coordination of experience obtained so as to incorporate features which showed the need or opportunity for improvement.

Principles of Design. Three general principles which had great influence on the design of the Presidents' Conference Committee trucks from the noise and vibration standpoint were as follows:

(*a*) It was believed that much of the noise and high frequency vibration experienced in older types of street car originated in the solid steel wheels running on very rigid track and in loose truck parts, particularly in cases where the looseness and play was likely to increase rapidly with wear. In the latter case, the campaign to eliminate loose parts was of course directed to reducing maintenance expense as well as noise and vibration of the parts affected.

(*b*) It was found that in many of the older types of street car truck the truck frame and spring system transmitted considerable vibration to the car body and an attempt was made to design a spring system which would transmit as little objectionable vibration as possible.

(*c*) It was noticed that many street cars had a very objectionable

fore-and-aft vibration, particularly during starting and stopping, and it was believed that this was due partly to roughness of the electrical control and partly to the existence of longitudinal flexibility in the car body and trucks where no flexibility was needed. This flexibility allowed longitudinal vibrations to be built up during acceleration and braking of the car. Attempts were made both to increase the smoothness of the control and to decrease the longitudinal flexibility to the point where vibrations did not occur.

Wheel Test Machine. This machine was arranged so that a single car wheel could be operated on a circular track. The wheel was mounted on a stub axle at one end of a horizontal arm, the other end of which was attached to a rotating driving head in the center of the track. The arm was very large and rigid so as to be almost free of vibrations under operating conditions.

The driving head was rotated by a 25 h.p. variable speed motor driving through a flexible coupling and a worm gear speed-reducing unit.

The outer end of the rotating arm was supported on the stub axle on which the wheel was mounted. The axle was arranged to take standard $3\frac{1}{4} \times 6$ in. railway journal bearings or the corresponding size of roller bearings.

The diameter of the track was 25 ft. and a maximum speed of 33 m.p.h. was provided for.

Both the central driving head and the track were supported on very heavy reinforced concrete in order to simulate a monolithic roadbed, to avoid vibrations due to the machine itself and to resist the large vertical and centrifugal forces resulting from operation at the higher speeds under heavy loads.

The rail was so arranged that the gap at one place was $\frac{1}{16}$ in., at another $\frac{1}{8}$ in. and at a third place $\frac{1}{4}$ in. In this way, the phenomena associated with operation over different types of joints could be studied. Rail ends were tied together by means of 6 hole joint plates and each joint was supported by a tie directly beneath it.

It was found difficult to maintain rail heads and joints in uniform condition so that successive tests would be exactly comparable, and rail and joints were reconditioned at frequent intervals during the tests to reduce variations as far as possible.

The load carried by the wheel was varied by placing weights in a box on the rotating arm. In most cases the maximum load was 7000 lb. at the wheel, although in a few tests loads as high as 10,000 lb. were employed.

Great precautions were taken in soundproofing the equipment and in insulating moving parts from the foundations. In this way all noise except that caused by the rolling of the wheel was reduced to a negligible value except that there was still some noise due to windage of the rotating parts.

Provisions were made for measuring the static load deflections of resilient wheels for radial, transverse and torsional loading. In addition, the radial deflections under rolling conditions were determined by means of a scribe fastened to one member and scratching upon a metal plate attached to another part.

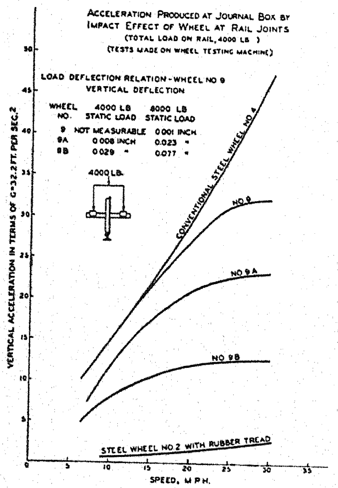


FIG. 85

Fig. 85 shows vertical accelerations of the journal boxes for different speeds of rolling and for a number of different wheels. For each wheel and each speed the results represent the average of the maximum values obtained while rolling around the circular track; that is, at the widest joint.

The upper curve shows a conventional steel wheel for which it will be noted that accelerations approaching 50 times gravity were measured at speeds of 30 m.p.h. Wheels 9, 9A and 9B are the same resilient wheel in which the tire member is supported from two parallel jig plates by resilient disks of circular form. In the case of #9 the disks were made of Asbesto-metallic friction block so as to obtain as little resilience as possible. Wheel 9A is the same wheel with disks made of 95° Durometer rubber. Wheel #9B is the same wheel with disks made of 72° Durometer rubber for which the wheel was originally designed. The curves show the remarkable results obtained in

decreasing the acceleration of the journal box by increasing the resilience and increasing the flexibility of the wheel.

The lowest curve was obtained by using a steel wheel similar to #4 with a piece of rubber belting cemented to the tread. It serves to indicate the results which might be obtained if even solid rubber tires could be used on a street car wheel.

All these wheels had substantially the same nominal tread diameter so that the variations shown are very nearly accurate measures of the effects of flexibility in the wheel.

Fig. 86 is a general summary of results obtained for a number of different resilient wheels. The wheels were run under vertical loads of 4000 lb. and the average journal box acceleration at 10, 20 and 30 m.p.h. has been plotted against the static radial strain for a vertical load of 8000 lb.

The wheels are not all of the same size and it will be noticed that wheel #8, which had a diameter of $28\frac{1}{2}$ in., showed a lower acceleration than the 26-in. wheels, and wheels #14 and 14A, with a diameter of 24 in., had higher accelerations. This is of course in the direction which would be expected since a large diameter wheel should pass more easily over a fixed rail joint than a wheel of small diameter.

In the first cars which were built after these tests, wheels were used which had a static radial deflection of about 0.14 to 0.19 in. under 8000 lb. which is definitely below the sharp up-turn in the test curve. This was found to give a perceptible shudder in the very rigid car structure when passing over special work. Later tests were made with wheels having about 0.60 in. static radial deflection under 8000 lb. load and the shudder of the car body was found to disappear entirely. It is probable that the performance of these wheels would correspond to a vertical acceleration of the journal box equal to gravity under the conditions shown in Fig. 86.

Fig. 87 shows the results of tests of a number of different designs of wheel on the wheel testing machine. Wheels *A* and *B* were among those which different inventors were attempting to market at the time. Wheel *C* was one of the first experimental wheels, which had good cushioning effects but which was not sufficiently rugged to stand up in service.

Considerable work was done on the noise of rolling wheels. A

microphone, which picked up the noise, was placed about 6 ft. from the track and so arranged as not to be affected by air currents produced by a passing wheel.

Fig. 88 shows the results for various speeds for the same wheels as in Fig. 85.

It will be noted that the most resilient wheel was of the order of 8 decibels quieter than the solid steel wheel.

Attempts were made to analyze the effect on noise of a large number of variables which existed during the test. There were not enough

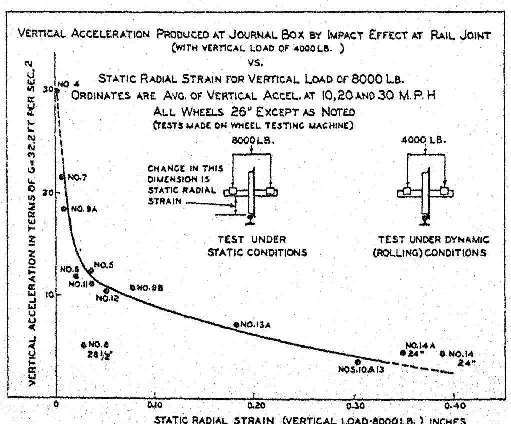


FIG. 86

observations to allow this to be done with any assurance, but in order to indicate the results obtained for what they are worth, Fig. 89 is included which may be of interest in showing the general effect of a number of factors. The symbols used are as follows:

DB = Decibels above 0.001 dynes per sq. cm. for an equally loud 1000-cycle tone.

W_S = Spring-borne weight in lb. of the resilient wheel under test; that is, journals, axle and wheel less the tire member.

S_L = The lateral deflection in the wheel produced by the action of a 2000-lb. force on the side of a tire measured in thousandths of an inch.

S_F = Radial deflection under 8000 lb. load measured in thousandths of an inch.

A = Surface area of tire member exposed to the outside air, measured in sq. in.

m = Number of rubber-insulated rings in a composite tire (if used) counting the flange as a ring but not counting the main part of the tire in which the rubber-supported rings are held.

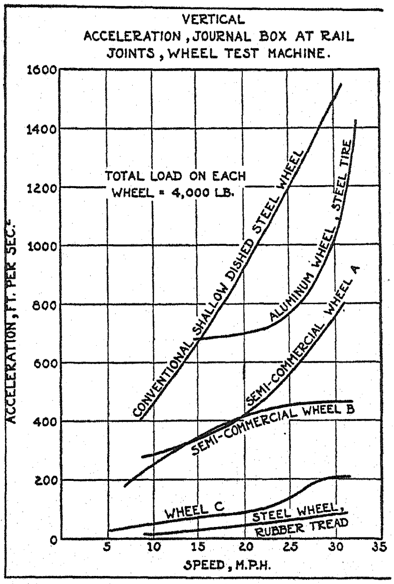


FIG. 87

t = Fractional part of the load carried by rubber in shear, expressed as a decimal fraction.

r = Shear stress in rubber produced by a 4000-lb. static radial load, expressed in lb. per sq. in.

v = Fractional part of the load carried by rubber in compression, expressed as decimal fraction.

u = Compression stress in rubber produced by a 4000-lb. static radial load, expressed in lb. per sq. in.

Another test made on the wheel-testing machine was a wedge jump test which gave further insight into the behavior and functions of

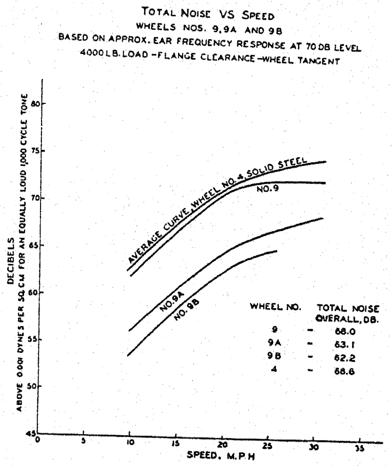


Fig. 88

resilient wheels. These tests were made by fastening a steel wedge on the surface of the rail and running the wheels over the wedge so

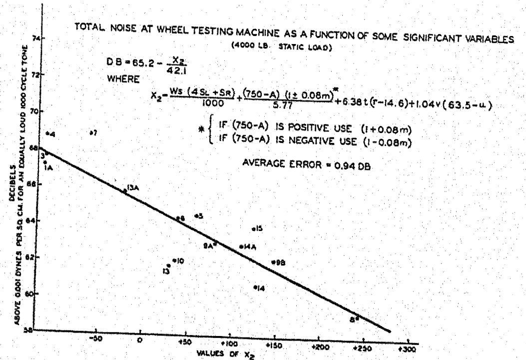


Fig. 89

that they jumped off the thick end back to the normal rail level. The wedge was a piece of steel 15 in. long. At the thin end it was $\frac{1}{64}$ in.

thick. It increased gradually up to $\frac{3}{32}$ in. in a distance of 3 in. and retained this thickness to the other end. The thick end was hardened so that conditions might remain essentially the same throughout the tests.

In these wedge jump tests it was found that above a certain critical speed, the speed made no appreciable difference in the vertical acceleration measured at the journal box when the wheel hit the track after leaving the thick end of the wedge. This critical speed appeared to be the one at which the tire member ceased to roll off the edge of the wedge and actually took on a trajectory.

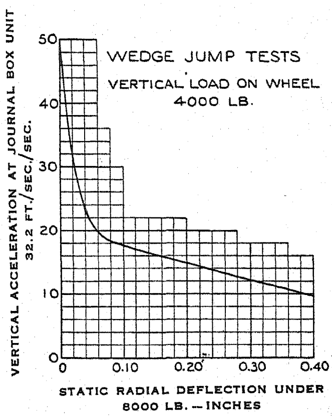


FIG. 90

Values of maximum vertical acceleration were obtained with a large number of different wheels and the results are reasonably well represented by Fig. 90.

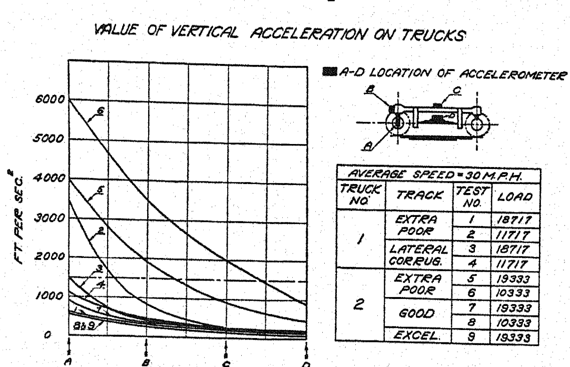
Observations were also made by placing white paper on the rail head beyond the wedge and noting the marks made on the paper by the tire of the wheel. In the case of the solid wheel there were from three to six or seven distinct impressions on the paper, indicating a succession of jumps. That is, after leaving the wedge, the wheels struck the rail, jumped up in the air, struck the rail, jumped up again, etc. In one case at a speed of 10 m.p.h., the whole horizontal distance between the point struck after leaving the wedge and the point struck after the first bounce was 2.0 in. This was followed by a bounce of 1.3 in., and this by a bounce of 0.8 in. At the same speed, resilient wheels gave first distances of from 0.5 to 1.5 in. and in most cases a bounce and second impression could not be discovered.

Elimination of Loose Parts. Rubber was used to a considerable extent in the design of PCC trucks to allow motion between the parts without the possibility of wear or rattle, and also to provide insulation between the wheels and the car body, which would be effective in

preventing the transmission of noise and high frequency vibration. Rubber springs were used in place of the commoner steel springs and other moving parts were mounted in rubber wherever any possibility of motion existed.

Spring Suspension. Before the development of the PCC truck, extensive investigations were made of the spring suspension systems of existing car trucks. Accelerometers were used to measure vertical acceleration of different parts of the truck structure.

In Fig. 91 the results of some of these tests are shown, all measurements being at a speed of about 30 m.p.h.



AVERAGE SPEED = 30 M.P.H.			
TRUCK NO.	TRACK	TEST NO.	LOAD
1	EXTRA POOR	1	18717
		2	11717
	LATERAL CORRUG.	3	18717
		4	11717
2	EXTRA POOR	5	19333
		6	10333
	GOOD	7	19333
		8	10333
	EXCEL.	9	19333

FIG. 91

The diagrammatic representation of the truck in Fig. 91 is for illustrative purposes only and does not represent accurately the structure of either of the trucks which were actually tested.

The curves in Fig. 91 show the vertical accelerations in a number of tests at the four points, *A*, *B*, *C*, and *D*. Point *A* is taken directly on the journal box and, of course, shows the highest maximum acceleration. Point *B* is taken on the truck frame directly above the journal box, and results show that the journal box springs reduce the measured acceleration to about one half of what it is at the journal box. Point *C* is taken in the middle of the truck frame. The acceleration here is again reduced about in half Point *D* is on the swing bolster above the bolster springs, and again the reduction in acceleration is roughly of the order of one half.

Tests were made on a large variety of truck springs of different types, assembled in different ways. One of the most interesting observations concerned leaf springs such as are commonly used for bolster springs. It was known, of course, that the friction of these springs tends to make them hard when subjected to high frequency impulses, but the tests showed that this effect was present to a very remarkable degree.

In one test the vertical acceleration of the car body was measured and indicated a force which would have deflected the spring statically about $3\frac{1}{4}$ in. The actual spring deflection at the time was only about $\frac{3}{4}$ in. Evidently under these conditions the spring has comparatively little value, and it was concluded that a very large part of the high frequency vibrations which were transmitted into the conventional car bodies were transmitted by the leaf springs, which behaved almost as if they were solid. The valuable damping action of the friction in the leaf spring is, therefore, obtained at the expense of the almost perfect transmission of high frequency vibration.

In order to avoid difficulties with leaf springs, rubber springs were used in the design of the PCC truck. Even in the case of rubber, it was found that accelerations of the car body were produced, which might be from 5 to 8 times the accelerations which would be calculated from measurements of the spring deflection and from the static characteristics of the springs. This led to the gradual adoption of softer and softer springs, particularly in the resilient wheels. The first commercially designed PCC truck had comparatively stiff springs and wheels, in view of the radical nature of the design. In later cars wheels in particular have been made two or three times as flexible, with a marked reduction in noise and shiver when passing over special work and bad joints.

One of the principles embodied in the design of the PCC truck was that so far as possible a number of springs in series should be avoided. In many other truck designs there will be, for example, journal box springs and bolster springs, and the truck frame will act as a mass supported by these two springs. In this position, the truck frame or other sprung mass can pick up comparatively high frequency vibrations, the energy of which has to be dissipated in shocks or in slower vibration of the car body. In order to avoid this condition, the PCC

cars have only journal box springs, in addition to the flexibility in the wheels, and there are no bolster springs.

It is interesting to note that in some cases designers have gone to the opposite extreme and have built trucks, particularly for railroad service, which have two or three sets of bolster springs in series with small masses in between the springs. It seems probable that such a multiple spring system does not tend to pick up any one vibration,

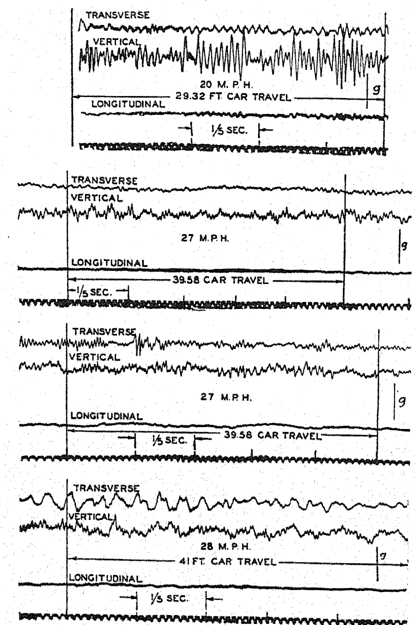


Fig. 92. Conventional Cars — Good Rail

but that the different kinds of possible vibrations quickly interfere with each other. Thus a similar result may be obtained by using one spring or by using a considerable number, whereas the system with two springs in series and a mass between has often caused objectionable vibration.

Car Body Vibration. The records reproduced in Fig. 92 show the characteristic floor vibrations of four different conventional cars on

good track at speeds from 20 to 28 m.p.h., as indicated on the records. Transverse vertical and longitudinal vibrations were recorded, the time intervals are marked, and a measure of the accelerations can be obtained from the vertical line labeled "g" in each record. The length of this line represents an acceleration of 32.2 ft./sec./sec.

For comparison, the records of Fig. 93 show the performance of an experimental PCC car on the same rail and those of Fig. 94 show the

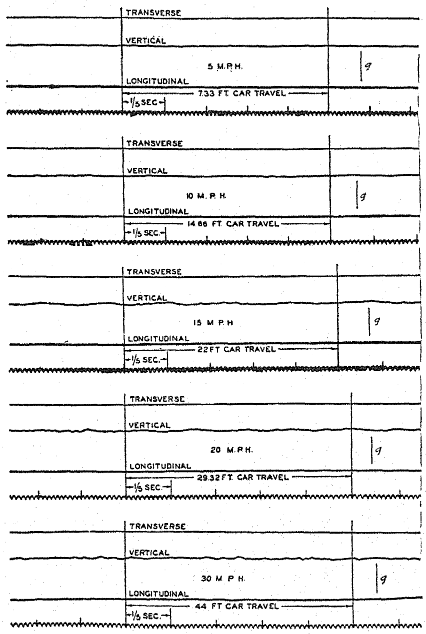


FIG. 93. P.C.C.-MODEL "B" CAR — GOOD RAIL

performance of the same car on very poor rail. The last two sets are made at speeds from 5 to 30 m.p.h., as marked on each record.

The records in Fig. 95 show the performance of one of the first commercial PCC cars on good rail at 20 m.p.h., on poor rail at 20 m.p.h., and good rail at 30 m.p.h. The performance is not quite so good as that of the experimental car, but is still remarkably better than the performance of the conventional cars shown in Fig. 92.

Starting Vibrations. A subject on which considerable work was done by the PCC engineers was the rapid starting and stopping of cars, without objectionable effect on the passengers. It was found that on many conventional cars there was longitudinal flexibility, which served no good purpose and which allowed longitudinal vibrations to be set up during starting or braking. One source of this flexibility was the usual truck design, which placed the center pin well

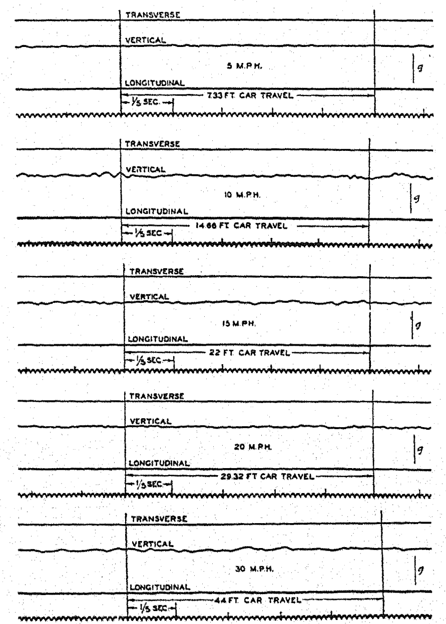


FIG. 94. P.C.C.-MODEL "B" CAR — POOR RAIL

above the level of the axles, so that a longitudinal force on the car body tends to tilt the truck and permit the car body to move fore and aft slightly, due to the height of the center pin. In the PCC design, a special long kingpin was used, with the object of keeping the longitudinal force on the car body at or below the height of the axles. By this means tilting of the truck during starting and stopping is greatly reduced, and the objectionable vibrations cannot be sustained.

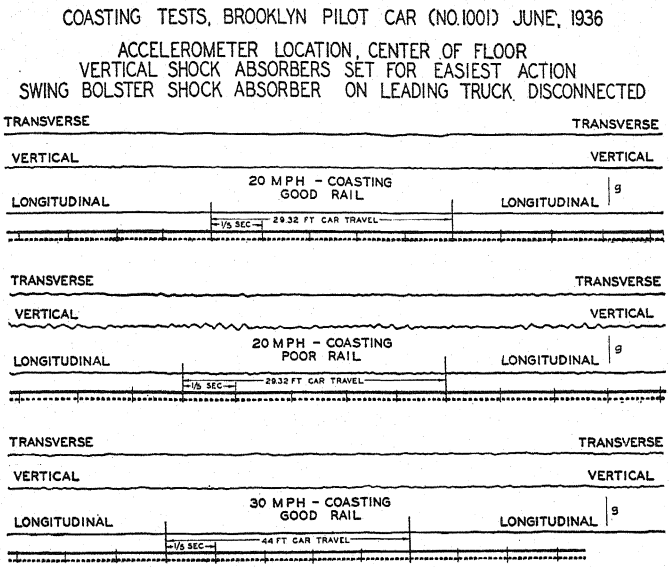

COASTING TESTS, BROOKLYN PILOT CAR (NO.1001) JUNE, 1936
ACCELEROMETER LOCATION, CENTER OF FLOOR
VERTICAL SHOCK ABSORBERS SET FOR EASIEST ACTION
SWING BOLSTER SHOCK ABSORBER ON LEADING TRUCK DISCONNECTED
TRANSVERSE
VERTICAL
LONGITUDINAL
20 MPH - COASTING
GOOD RAIL
1/5 SEC
29.32 FT CAR TRAVEL
20 MPH - COASTING
POOR RAIL
30 MPH - COASTING
GOOD RAIL
44 FT CAR TRAVEL
g

FIG. 95

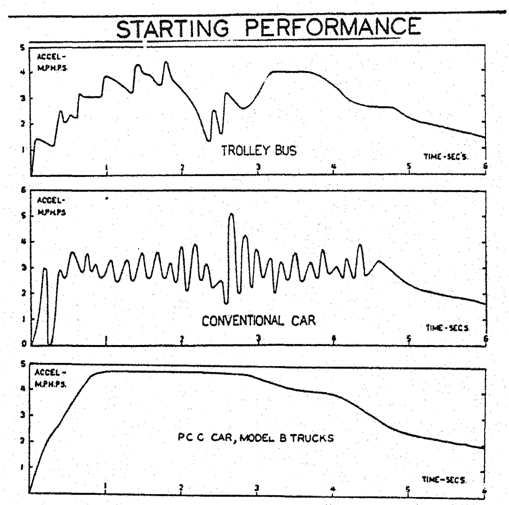

STARTING PERFORMANCE
ACCEL-
M.P.H.P.S.
TROLLEY BUS
TIME-SEC'S.
CONVENTIONAL CAR
TIME-SECS.
P C C CAR, MODEL B TRUCKS
TIME-SECS.

FIG. 96

Other work attacked one direct cause of the vibrations, which is variation in tractive effort or braking. Fig. 96, for example, shows in the upper record starting performance of a trolley bus fitted with automatic control, as available before the PCC development. The result is typical of either trolley bus or street car controls with the control functioning correctly. The middle record shows the effect of varying tractive effort on a car with longitudinal flexibility, where the

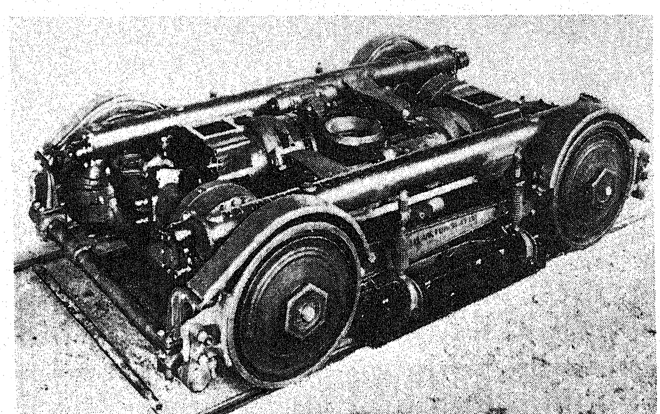


FIG. 97. TRUCK FOR P.C.C. TYPE CAR

period of fore-and-aft vibration was more or less resonant with certain steps in the control. This record was obtained from a commercial car and is not in any way exceptional. The lower record shows the performance of an experimental control, and commercial street car and trolley bus controls were developed to give substantially the same result.

PCC Truck Design. Fig. 97 shows a complete PCC truck with flexible wheels, rubber journal box springs, motors with right angle drive, swing bolsters without springs, provision for long king pin, and general use of rubber mountings. The design of the truck has been varied in detail, according to different conditions and as experience has been obtained which would allow improvements.

Other Truck Designs. A number of other designs have been built to accomplish the same general purpose as the PCC truck, but

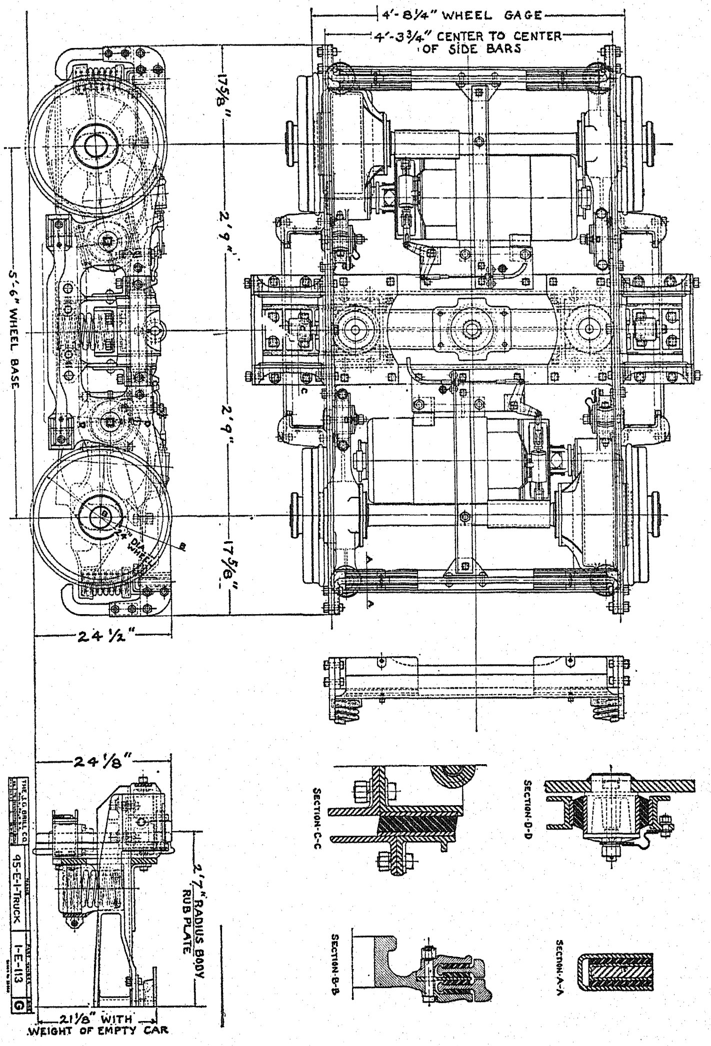

4'-8¼" WHEEL GAGE
4'-3¾" CENTER TO CENTER OF SIDE BARS
17⅝"
2'9"
2'9"
17⅝"
5'-6" WHEEL BASE
24½"
24⅛"
2'7" RADIUS BODY RUB PLATE
21⅛" WITH WEIGHT OF EMPTY CAR
SECTION-C-C
SECTION-D-D
SECTION-B-B
SECTION-A-A
THE J. G. BRILL CO.
45-E-1-TRUCK
1-E-113
G

FIG. 98

using a different means. An example is a street car truck built by the J. G. Brill Company (Fig. 98).

This truck has flexible wheels, but of entirely different design from those used in the PCC truck. Rubber is also used in a number of other parts of the truck, as shown in Fig. 98. The main truck springs are helical steel, and the bolster is spring supported. The center plate and motor arrangement are more nearly conventional than on the PCC truck.

PART IV. RAILROAD VEHICLES

Chapter 16

RELATION BETWEEN WHEELS AND TRACK

Introduction. Chapters 16 and 17 will investigate some of the characteristics of vehicles which have flanged wheels and run on rails, particularly railroad locomotives and cars. There is always some clearance between the wheel flanges and the rail. The wheels tend to oscillate back and forth through this clearance, causing oscillations of the superstructure which may become serious. On straight, or tangent, track an oscillating vehicle is guided by impacts between the various wheel flanges and the rails. The vehicle can take any position within the limits of the flange clearance and is kept within these limits by a succession of flange impacts. On curved track, however, the vehicle generally takes up a position in which one or more wheel flanges are in contact with rails and remain there, guiding the vehicle steadily round the curve. This position may become unstable, in which case the vehicle will oscillate and cause blows between the wheel flanges and the rail.

Clearances and Wheel Contours. The standard wheel contours used on American railroads allow a clearance of approximately $\frac{3}{16}$ in. between the flange and the rail when an axle is placed symmetrically on track of $56\frac{1}{2}$ in. gauge, the wheels being spaced $53\frac{3}{8}$ in. back to back. This gives a total clearance in the track of approximately $\frac{3}{8}$ in.

On some railroads the track gauge has been reduced to $56\frac{1}{4}$ in. in order to reduce the clearance and thereby reduce oscillations of equipment. On some locomotives, some wheels are spaced less than $53\frac{3}{8}$ in. to as low as $53\frac{1}{8}$ in. back to back, in order to reduce flange pressures on curves.

As the track and wheels wear, the flange clearances increase until it becomes necessary to correct them.

Besides flange clearances, it is necessary to consider the other lateral clearances in railroad equipment, principally between the truck frames and the axles. These clearances occur between frames and journal boxes and in journal bearings and are conveniently referred to as journal box clearances, hub clearances or internal clearances. The total internal clearance of an axle is generally about $\frac{1}{4}$ in. when new and may increase to as much as about $\frac{3}{4}$ in. when the parts are worn. In some cases axles are designed to have a very large internal clearance, up to about $2\frac{1}{2}$ in. for special reasons which will be considered later.

To summarize the typical figures, which are useful in general studies, we have

Total flange clearance in track:

New wheels and rail	$\frac{3}{8}$ in.
Normal condition	$\frac{3}{4}$ in.
Worn condition	$1\frac{1}{4}$ in.

Total internal clearance:

New parts	$\frac{1}{4}$ in.
Normal condition	$\frac{3}{8}$ in.
Worn condition	$\frac{3}{4}$ in.

The majority of railroad wheels are coned; that is, the treads have a taper of 1 in 20 when they are new. Wear varies with the conditions, but the taper usually wears down and many wheel treads are worn hollow before they are corrected. In a few cases, the surface of the tread is rolled up towards the flange, giving the effect of increased taper with wear, but this is generally due to unusual track conditions, with excessive speed on curves.

Friction and Creep. The action of wheels in rolling on the surface of the rail does not require special study here as long as the motion is purely rolling. However, the ordinary motion of wheels is constrained by the circumstances that two or more axles are usually placed in one frame and thus forced to run parallel to each other and also by the effects of flange impacts in changing the direction of motion These result in the wheels having to slip slightly as well as to roll

This is most important and will be studied in some detail as it is the basis of further studies.

If a wheel slips considerably, as when going around a sharp curve, the coefficient of friction is generally assumed to be 25%, which has become a conventional figure. The actual value varies with conditions of the rail, etc. It may reach 35% or even higher on a good dry rail and may drop to 15% or less on a wet slippery rail. However, 25% represents a very fair average condition.

If a rolling wheel slips very slightly, so that the velocity of slip is a very small fraction of the velocity of roll, then another factor enters, which is the elasticity of the metal composing the wheel and the rail. It may be easier to visualize this with a rubber wheel, which is quite flexible. What is true of the rubber wheel is also true of the steel wheel, except in a lesser degree.

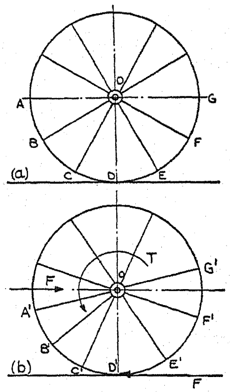


FIG. 99

Consider Fig. 99 (*a*) which represents a rubber wheel with a number of equally spaced radii OA, OB, OC, etc., drawn on the surface of the unstrained rubber. Now if the wheel is twisted by a turning moment T, which balances the moment due to the forces F, F in Fig. 99 (*b*), the wheel will appear as in the latter figure. The rubber is compressed on the side A', B', C', D', and extended on the side E', F', G', H'. Look at the small section CD in Fig. 99 (*a*) which is next to the area of contact between the wheel and the surface on which it rests. Suppose this represents $\frac{1}{n}$ of the perimeter of the wheel $= \frac{2\pi r}{n}$, where $r =$ the radius of the wheel. Then in $\frac{1}{n}$ of a revolution the point of contact will move from D to C and the wheel will roll a distance $CD = \frac{2\pi r}{n}\cdot$ Therefore, as is obvious, in one revolution it will roll a distance of $2\pi r$.

Now look at the same section of rubber in Fig. 99 (*b*), where it is marked $C'D'$. It is shorter due to being in compression. Therefore

in $\frac{1}{n}$ of a revolution the wheel only rolls a distance $C'D'$ which is less than CD; that is, less than $\frac{2\pi r}{n}$. Therefore in one revolution it will roll a distance less than $2\pi r$. This is the same as saying that when the wheel in Fig. 99 (*b*) rolls a distance $2\pi r$ it will turn more than one revolution. It will be evident that this extra turning is due to the fact that compressed rubber is rolling into the area of contact and the greater the compression the greater the extra turning. Since the compression of the rubber is due to the driving force, the extra turning increases with the driving force. It will therefore be evident that if a wheel rolls freely it turns through one revolution in a distance $2\pi r$, but if it is a driven wheel, it will turn through more than one revolution, the difference increasing with the driving force. The extra movement of the point of contact, over what it would be if the motion was pure rolling, is called the creep. The force required to produce a given creep is given by

$$\text{Force at the point of contact} = \text{coefficient} \times \frac{\text{velocity of creep}}{\text{velocity of roll}}.$$

The coefficient is called the creepage coefficient and usually represented by the letter f, it has the dimensions of a force, thus

$$\text{Force} = f \times \frac{\text{velocity of creep}}{\text{velocity of roll}} \quad . \quad . \quad . \quad . \quad . \quad (22)$$

A similar argument will show that if a rolling wheel is acted on by a force at right angles to the plane of the wheel, the material deflects in the direction of the force and rolls into the area of contact at a slight angle to the direction of rolling. As a result the wheel creeps slightly in the direction of the force and again the ratio of creep to roll increases with the lateral force so that equation (22) applies here also. The coefficients are not necessarily the same for creep in the direction of roll and creep at right angles to the direction of roll, but are generally assumed to be the same in practical applications. This assumption is made (*a*) because very little information is available at present as to the values of the coefficients, (*b*) because the results of most calculations are very little affected by considerable changes in

the numerical values of the coefficients and (*c*) because of the great simplification in formulae without any practical disadvantage as regards accuracy of results.

The theory of creep was developed by Dr. F. W. Carter, who has calculated the theoretical value of the creepage coefficient for the case of creep in the direction of rolling. Dr. Carter's formula is: *

$$f = 3500 \sqrt{\text{diameter of wheel (in.)} \times \text{weight on axles (lb.)}}$$

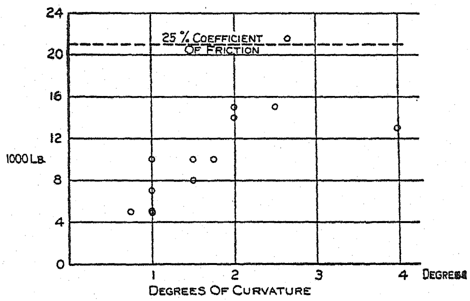


FIG. 100

The present author has observed actual values of the creepage force in practice by measuring the lateral force required to guide one axle of a high speed electric locomotive on curves. The wheels on this axle had blind tires, so that there was no flange force. The force measured was the total creepage force, and the ratio of

$$\frac{\text{velocity of creep}}{\text{velocity of roll}}$$

was measured by estimating the angle between the center line of the locomotive and the tangent to the track. The observations, while admittedly rough, are numerous and reasonably consistent. It is interesting to note that they check Dr. Carter's theoretical value of the creepage coefficient very closely. The author's observations are

* "The Running of Locomotives, with Reference to Their Tendency to Derail," F. W. Carter. *The Institution of Civil Engineers*, 1930. "On the Action of a Locomotive Driving Wheel," *Proceedings, Royal Society*, Series A, Vol. 112, 1926, p. 151.

summarized in Fig. 100 which covers speeds from 25 to 60 m.p.h. and in Fig. 101 covering speeds from 60 to 70 m.p.h.

There is no noticeable effect of speed. In these diagrams the creepage force is plotted against the degree of curvature. The ratio $\dfrac{\text{velocity of creep}}{\text{velocity of roll}}$ for this locomotive varies almost exactly with the curvature, hence the diagrams show that the creepage coefficient

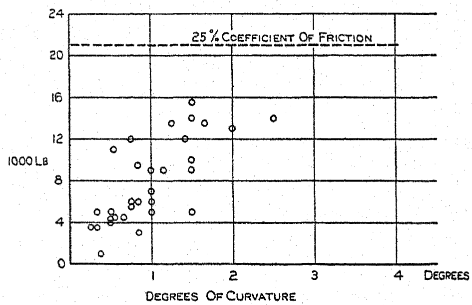


Fig. 101

is approximately a constant. On account of the close checks which the author has obtained between these observations and Dr. Carter's theory, the theoretical values may be used with considerable confidence. It will be evident that there is an upper limit to any creepage force. It cannot exceed the frictional force required to slip the wheel. Actually a little consideration will show that there is a gradual transition from creep to slip. Returning for a moment to Fig. 99 (*b*) it will be noted that the wheel material leaving the area of contact is in tension. The actual area of contact is shown in plan to an enlarged scale in Fig. 102. The area P represents compressed material which has rolled into the area. The tension on the back end of the area will cause a portion, represented by Q to slip, relieving the compression and putting the material in tension. As the tangential force acting on the area of contact increases, the section Q, which slips, increases until finally it accounts for the whole area of contact and the motion

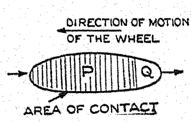


Fig. 102

becomes pure slipping against the force of friction. It follows from this discussion that it will be correct to use the creep theory for wheel motions for which the creep or slip is very small compared to the roll. It will be correct to use the slip theory for wheel motions for which the creep or slip is larger, and for which the frictional force due to slip is lower than the calculated creep force. As a matter of practical convenience either theory may be used often without any very great difference in the calculated results and under different conditions one theory or the other may prove the easier to handle and this may determine which one is actually used. It is fortunate that oscillations on straight track, for which the creep theory is most accurate, are often more easily handled by means of this theory. Conversely, motion on sharp curves, for which the slip theory is accurate, is best studied in the light of that theory. On high speed curves of long radius, the slip theory is generally used, but the reduction in the creepage force is to some extent recognized by the practice of assuming a lower coefficient of friction, generally 15%, for such cases.

Chapter 17

MOTION OF TRUCKS ON STRAIGHT TRACK

Considerable trouble is experienced from time to time with cars and locomotives nosing on straight track, either regularly or at irregular intervals at rough spots in the track. A number of very distinct motions are involved and it seems best to review the experience before going into any theory. Experience is partly a collection of general observations accumulated by many people over many years and partly the results of controlled experiments in which more or less accurate measurements are made. The general observations of nosing are numerous, but it has been difficult for observers to distinguish the many factors which may enter into nosing in different cases. Sufficient experimental measurements have, however, been made of the motions of different locomotives and the forces which they exert on the track to show very definitely some causes of nosing and to establish practical ways of avoiding it.

The most general fact is that most railroad vehicles behave in the same way each time they pass over a given piece of track. They hit the same places with the same force every time if the speed is the same and if internal variables such as the unbalanced forces in steam locomotives are absent.

Records of the behavior of electric locomotives show very remarkable constancy in that a record over a given piece of track can be almost exactly duplicated by successive runs. Then it is found in most cases that over a certain range of speed the vehicle hits the same places on the track independently of speed. As the speed increases the blows get harder but they occur at the same spots and additional blows appear. As the speed increases and the blows become more severe, they begin to shift with speed and when a continual oscillation takes place it has a very nearly constant time period; that is, the blows get farther apart as the speed increases. This regular period is

disturbed at unusually rough spots in the track at which the location of the blows remains the same. At these speeds there are three broad classes of vehicles. Those which run steadily, to the extent that no regular oscillation is built up; those which run steadily on very good track but oscillate regularly on poorer track; and those which oscillate regularly on any track, however good.

Some vehicles operate always in the range where the distance between blows is independent of the speed. When this happens the track often becomes sufficiently deformed to force this motion to persist. This applies particularly to systems in which dense traffic is made up chiefly of similar vehicles running at about the same speed as is the case on numerous suburban electrifications.

Tests and general experience on double-ended electric locomotives are instructive. If the locomotive consists of a rigid driving frame with guiding trucks at each end, the locomotive is generally:

(1) Unstable at moderate speed on all track when the leading truck has high resistance and the trailing truck low resistance.

(2) Unstable at sufficiently high speed on sufficiently rough track when the leading and trailing trucks have equal resistance.

(3) Stable on all but unusually rough track when the leading truck has low resistance and the trailing truck high resistance.

If the locomotive has articulated driving trucks without restraint, the locomotive will generally oscillate on any track if the speed is high enough.

On passenger cars it is found that good riding is generally associated with very flexible spring suspensions.

The effects of coning of wheel tires are also interesting. In general, coning has little effect on the nosing of large locomotives. On cars a difference is noted which is broadly between (*a*) cars with heavy bodies and light trucks and (*b*) cars with light bodies and relatively heavy trucks. In the first case, if the usual coning of tires is eliminated, a nosing truck may not nose so badly but is liable to oscillate against the rail, grinding and cutting the flanges. In the second case, nosing can sometimes be reduced or eliminated by using cylindrical tread wheels.

Finally, the height of the center of gravity of the vehicle is a factor which has caused much discussion. In conventional steam

locomotive design it has long been found that a high center of gravity (75–80 in.) was associated with good riding. In some electric locomotives there have been similar indications, but others and most cars are successfully run at high speed with quite low centers of gravity of the order of 50 to 65 in. above the rail.

A little consideration will show why rule-of-thumb methods which suit one class of vehicles often prove very misleading when an attempt is made to apply them more generally. The object of this chapter is to develop the theory of motion of vehicles so that the observed facts can be explained and the theory can be used as a tool in designing satisfactory riding vehicles or in correcting those which cause trouble. The theories developed below have been used successfully in these ways.

It may be well to emphasize that this chapter is concerned with operation on straight track. Vehicles must also run on curves and under other conditions and the results obtained here give only a partial picture of the behavior of any particular kind of vehicle.

A study of rail vehicles soon shows two factors which enter into oscillation, the track irregularities which tend to produce a forced oscillation and an inherent capacity of the vehicle to oscillate independently of the forced oscillation.

The forced oscillation is due generally to irregular surface of the track which causes both rolling and nosing.

The "inherent" oscillations are best considered from two separate viewpoints, that of force and that of energy. If an oscillation is to sustain itself, two conditions must be satisfied. First there must be forces which act on the vehicle in such a way that when it moves towards one rail its direction is changed back towards the other rail. If this is not so, the vehicle will merely oscillate in a minor way against one rail or the other. In the second place, if the vehicle does oscillate regularly it must in some way pick up enough energy to compensate for the inevitable losses due to slipping, spring friction, rail impact, etc.

The forces will be considered first. If a vehicle runs over to one rail, what will direct it back to the other? Suppose for simplicity the vehicle has two axles and runs over to the left rail. The front wheel flange will hit the rail and tend to turn the vehicle. Then the rear

wheel flange will hit and tend to turn the vehicle back against the left rail. If the vehicle is to turn definitely towards the right rail the blow on the front axle must be appreciably more effective in turning it than the blow on the rear axle. That is, the blow on the front axle must be either greater or at a greater distance from the center of gravity or both. This simple remark gives some indication at once of the behavior of several kinds of locomotive. If an otherwise symmetrical engine has a high resistance leading truck and a low resistance trailing truck, the leading truck will hit higher blows than the trailing truck and will be able to turn the engine from one rail to the other, helping to sustain an oscillation. If the high resistance truck is at the back the flange impacts at the back are higher and keep the engine from swinging away from a rail which it has just hit. This therefore tends to be a stable combination.

The same reasoning shows that, for steady operation on straight track, it is well to have a leading truck a short distance ahead of the center of gravity and a trailing truck a long way behind.

It will also be seen that in a short engine with a high center of gravity, blows on the flanges are largely absorbed by roll of the engine and hence the blows are small and have little effect in making the engine nose. For a short wheelbase engine, therefore, a high center of gravity is advantageous in reducing nosing. In a long engine or a car a flange blow near one end chiefly turns the vehicle, so that the height of the center of gravity is much less important.

We have just discussed one set of forces which affect oscillation; namely, the flange forces in relation to the center of gravity. There are three other classes of force which must be similarly considered, due to friction, coning of wheels and track irregularities.

Friction forces between the wheels and the rail act generally to keep the vehicle moving in the direction of its wheelbase. They resist side motion or rotation, but if a truck, after hitting one rail, becomes turned towards the other rail it is largely friction which will actually carry it across the clearance to the other rail. Friction therefore is an important item in any study of oscillation from one rail to the other. Friction always resists rotation of a truck and therefore some other forces must be present which are sufficient to overcome this friction before an oscillation can occur.

Forces due to coning of the wheels are also friction forces, but it will be seen that they act in the longitudinal direction instead of transverse to a truck. They are primarily turning forces which, whenever a truck moves towards one rail, tend to turn it back towards the other. They are therefore a source of oscillation. On cars they are important. On large locomotives, however, experience shows that the oscillations are practically independent of coning and that changes from tires worn hollow to tires with new coned contours do not produce any major change in the behavior. For this reason it is often preferable to omit the effect of coning entirely in calculating the oscillations of large locomotives. The omission simplifies the work and avoids the danger of drawing elaborate conclusions which depend on the exact amount of coning of the wheels and which are not borne out by experience. It should not be forgotten, however, that coning produces the important turning forces on a truck. If other turning forces, due to flange impacts, etc., are present, the effect of coning may be negligible, but if there are no other turning forces then coning should be carefully studied to see whether it will result in oscillation.

Forces due to track irregularities are not very complicated and the following principles cover most of the conditions which are met. Most oscillations are caused by poor track surface. When there is a local warping of the track surface the vehicle tends to roll as it passes over the rough spot. If it rolls, the center of gravity must move sideways and therefore the wheels must exert a side pressure on the track. Unless this side pressure is exactly in line with the center of gravity, it will exert a turning moment, possibly causing the vehicle to nose. There is another minor effect: suppose the left rail is low—when the front axles of the vehicle pass over the low spot, they will tend to slide down towards the left thus turning the vehicle towards the left rail and starting an oscillation.

Poor line, such as a kink in one rail, is not generally important unless the kink occurs at a spot where wheel flanges normally hit the rail due to some previous rough spots. In this case the kink is made worse by successive blows and aggravates the rebound of the wheels after the flanges strike. An isolated kink inwards normally will be straightened out by accidental flange impacts. An isolated kink outwards will not produce any effect. For the reasons outlined here,

"rough track" will generally be represented by assuming the surface to be warped.

Having discussed briefly the forces which act on a vehicle, let us consider the energy changes. It may be assumed that the speed is constant so that the only energy changes will be in the vibration energy. The most important thing is to look for ways in which the vibration energy can increase, because if there is no source of energy any vibration will be quickly damped out.

First consider flange impacts. If the rail and wheel are uniformly flexible, no energy can be gained through any flange impact. Some of the kinetic energy of motion is converted into strain energy in the rail, the wheel, etc., and is ultimately given back as kinetic energy of rebound or lost in friction, impact, etc. It follows that in all energy studies it is unnecessary to consider flange impacts except as a possible loss. This indicates at once how calculations can be simplified when the energy method can be used.

Secondly, we have friction forces between the wheels and the rail. It is often supposed that all friction results in loss of energy, but this is deceptive. It applies of course to the system as a whole, but does not necessarily apply to the vibration energy considered separately. Actually friction may be a medium by which the vibration energy is increased at the expense of the driving equipment of the vehicle.

Because of the importance of understanding this power of friction to increase vibration energy, let us consider the simple case of a 2-axle car O which, after hitting rail XX is in the position shown in Fig. 103. The impacts have turned the car towards rail YY but have been insufficient to give it any lateral velocity. That is, all the vibration energy has been lost in flange impacts and friction. Now the car is moving along the track with speed v and its inertia would carry it in the straight line OP. If it moves along OP, however, the wheels must slip sideways as well as roll and friction will oppose the slipping. As a result the car will slip and roll along a curved path until it moves in the direction in which it can roll without slipping, after which it rolls along a straight line towards the rail YY. The path is shown by OP'. The car has thus acquired a lateral velocity and

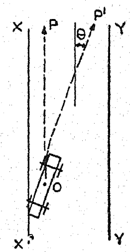


FIG. 103

corresponding kinetic energy through the medium of friction. This is vibration energy, as will be seen by considering that it must be absorbed in some other form when the car hits the rail YY. If the speed is v and the inclination of the car is θ the lateral velocity acquired is $v\theta$ and the vibration energy is

$$\frac{1}{2}\frac{W}{g}(v\theta)^2$$

which has been obtained from the energy of forward progression and ultimately from the propulsion equipment of the car.

Most other motions are against friction rather than with friction and result in losses of vibration energy. The friction due to slipping or creeping of coned wheels, however, will generally show a gain of energy. Whenever the axles turn in the direction which coning would make them turn, the coning does work and the vibration energy increases.

Finally, rough track causes motion and hence increases vibration energy except in obvious particular cases where the roughness happens to be such as to reduce an existing oscillation.

In energy calculations account must be taken of losses due to friction in springs, wearing surfaces, dampers, etc.

Motion of a Pair of Coned Wheels. If an axle with a pair of coned wheels rolls along a straight track, starting in some direction at a slight angle to the direction of the track, the coning will cause the axle to oscillate back and forth across the center line of the track and the path will be a sine curve, repeating in a distance

$$6.28\sqrt{\frac{ra}{\lambda}}$$

where r = radius of the wheel
$a = \frac{1}{2}$ the distance between rolling contacts of the wheels on the two rails, usually $\frac{1}{2} \times (\text{gage} + 2\frac{1}{2} \text{ in.})$
λ = coming of the wheels; that is the slope of the tread, usually $\frac{1}{20}$.

To prove this, consider Fig. 104. O is the center of the axle and A and B are the points of contact between the wheels and the rails. The

distance between these points is $2a$ and the coming of each wheel is λ. The position of the axle at line t is given by the distance y of its center to the right of the center line of the track and by the angle θ which its direction makes with the direction of the track.

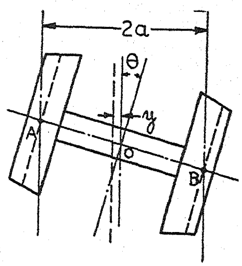


FIG. 104

If the radius of each wheel, measured at a distance a from the center of the axle, is r, then the radius at A is $r - \lambda y$ and at B is $r + \lambda y$.

Hence if the axle rolls a small distance ds (average of the two wheels)

Point A will roll $$\frac{r - \lambda y}{r} ds,$$

Point B will roll $$\frac{r + \lambda y}{r} ds,$$

Therefore, B will roll a distance $\dfrac{2\lambda y}{r} ds$ farther than A.

Therefore, $$\frac{d\theta}{ds} = -\frac{2\lambda y}{r} \times \frac{1}{2a} = -\frac{\lambda}{ra} y$$

But $$\theta = \frac{dy}{ds} \quad \therefore \quad \frac{d^2y}{ds^2} = -\frac{\lambda}{ra} \cdot y$$

A solution of which is:

$$y = y_{\max} \sin \sqrt{\frac{\lambda}{ra}}\, s$$

which repeats in a distance $2\pi \sqrt{\dfrac{ra}{\lambda}}$

Example. For a standard 36-in. diameter passenger car wheel with 1 in 20 coning on standard gage track,

$$\lambda = \tfrac{1}{20}$$

$$r = 18 \text{ in.} = 1.5 \text{ ft.}$$

$$a = \tfrac{1}{2}(56\tfrac{1}{2} + 2\tfrac{1}{2}) = 29.5 \text{ in.} = 2.46 \text{ ft.}$$

The distance in which the motion repeats is

$$6.28 \sqrt{\frac{1.5 \times 2.46}{(\frac{1}{20})}} \text{ ft.} = 54 \text{ ft.}$$

A larger wheel will roll in a longer wave.

On a curve of radius R ft. measured to the center line of the track, the ratio of the length of outer rail to length of inner rail is

$$\frac{R + a}{R - a}$$

The axle will roll round such a curve without slipping if the radii of the wheels are in the same ratio, that is if

$$\frac{R + a}{R - a} = \frac{r + \lambda y}{r - \lambda y}$$

The solution of this is $$y = \frac{ar}{R\lambda}$$

or, for 36-in. wheels, $\frac{1}{20}$ coning and standard gage,

$$y = \frac{2.46 \times 1.5}{R \times (\frac{1}{20})} \text{ ft.} = \frac{886}{R} \text{ in.}$$

where R is the radius of the curve in feet.

If the track clearance is limited by flanges to $\frac{3}{8}$ in. either side of center line, this will limit the curves around which coning will guide the axle freely to those of 2362 ft. radius or longer. That is about $2\frac{1}{2}°$.

It is easy to prove that this steady motion around a curve is stable and that if the axle is displaced, it will oscillate steadily around the curve with the same periodic distance as on straight track.

Motion of a Series of Axles in One Wheel Base. In most trucks and locomotive frames, a number of axles are forced to move together and can no longer move as they would individually. Such a rigid wheel base moves like a single axle in a sine curve, but with a longer periodic distance,

$$\text{Periodic distance} = 2\pi \sqrt{\frac{\Sigma(fl^2) + a^2\Sigma(f)}{a\lambda\Sigma\left(\dfrac{f}{r}\right)}}$$

where f = creepage coefficient for one axle
r = radius of a wheel
l = distance of an axle from the origin, which is the center of gravity of the f's
a = $\frac{1}{2}$ the track gage between wheel contacts
λ = coning of the wheels
Σ = represents the summation over all the axles ($\Sigma fl = 0$)

At the center of the wheelbase (as defined above, so that $\Sigma fl = 0$), the lateral force resisting motion is

$$\left[\frac{1}{v}\frac{dy}{dt} - \theta\right] \times \Sigma(2f)$$

and the moment resisting motion is

$$\frac{1}{v}\frac{d\theta}{dt}[\Sigma(2fl^2) + a^2\Sigma(2f)] + a\lambda y\Sigma\left(\frac{2f}{r}\right) + a\lambda\theta\Sigma\left(\frac{2fl}{r}\right)$$

where v = the velocity of motion
y = the lateral motion of the center of the wheelbase
θ = the angular motion of the wheelbase.

These results are obtained below.

Let x be measured in the direction of motion. Let y be measured laterally, to the right, in Fig. 105. Let the origin of coordinates move steadily down the track with the same speed as the truck. When the truck is central and symmetrical the origin will coincide with some convenient point O on the center line of the truck.

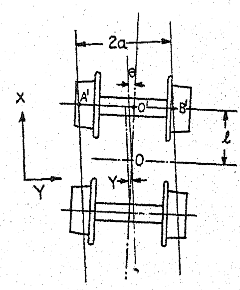


FIG. 105

Then if the point O is a distance y to the right and the truck center line is rotated through an angle θ, the velocity of the point of contact of a wheel A' is $v + a\dfrac{d\theta}{dt}$ in the direction of the center line of the track, and $\dfrac{dy}{dt} + l\dfrac{d\theta}{dt}$ at right angles to the center line.

If the axle $A'B'$ is rotating about its axis with angular velocity ω, the rolling velocity of the point of contact in the direction of the *truck* center line is

$$\left\{r - \lambda(y + l\theta)\right\}\omega$$

which is made up of

$\left\{r - \lambda(y + l\theta)\right\}\omega \cos\theta$ in the direction of the *track* center line and $\left\{r - \lambda(y + l\theta)\right\}\omega \sin\theta$ at right angles.

Now the velocity with which the point of contact creeps is the total velocity minus the rolling velocity. Therefore, the velocity of creep in the x direction is

$$v + a\frac{d\theta}{dt} - \left\{r - \lambda(y + l\theta)\right\}\omega$$

Since $\cos\theta = 1$ is a sufficiently close approximation, and in the y direction, the velocity of creep is

$$\frac{dy}{dt} + l\frac{d\theta}{dt} - \left\{r - \lambda(y + l\theta)\right\}\omega\theta$$

putting θ in place of $\sin\theta$.

Then the creepage force which must be overcome is

$$f \times \frac{\text{velocity of creep}}{\text{velocity of roll}}$$

where f = the creepage coefficient.

∴ Creepage force in the x direction is

$$f\left[\frac{v + a\dfrac{d\theta}{dt}}{r\omega\left\{1 - \dfrac{\lambda}{r}(y + l\theta)\right\}} - 1\right]$$

$$= f\left[\frac{v}{r\omega}\left(1 + \frac{a}{v}\frac{d\theta}{dt} + \frac{\lambda y}{r} + \frac{\lambda l\theta}{r}\right) - 1\right] \text{ approx.}$$

and creepage force in the y-direction is

$$f\left[\frac{\dfrac{dy}{dt}+l\dfrac{d\theta}{dt}}{r\omega\left\{1-\dfrac{\lambda}{r}(y+l\theta)\right\}}-\theta\right]=f\left[\frac{\dfrac{dy}{dt}+l\dfrac{d\theta}{dt}-r\omega\theta}{r\omega}\right] \text{ approx.}$$

the approximations being obtained by neglecting squares and products of y, θ and $\dfrac{d\theta}{dt}$, which are very small.

The forces at the opposite wheel, B', are obtained by changing the signs of a and λ. Adding the forces at the two wheels the total creepage forces on the axle are

In the x-direction, longitudinally

$$2f\left[\frac{v}{r\omega}-1\right]$$

In the y-direction, laterally

$$2f\left[\frac{\dfrac{dy}{dt}+l\dfrac{d\theta}{dt}-r\omega\theta}{r\omega}\right]$$

The moment about the point O is made up of

Total creepage force in the y-direction $\times\, l$

$+$ Creepage force at A' in x-direction $\times\, a$

$-$ Creepage force at B' in x-direction $\times\, a$

$$=2fl\left[\frac{\dfrac{dy}{dt}+l\dfrac{d\theta}{dt}-r\omega\theta}{r\omega}\right]+2fa\frac{v}{r\omega}\left[\frac{a}{v}\frac{d\theta}{dt}+\frac{\lambda y}{r}+\frac{\lambda l\theta}{r}\right]$$

In the most important case there is no longitudinal force; that is, the tractive or braking effort is neglected. Hence $v = r\omega$. Also, for simplicity, choose the origin of coordinates at a point which is the center of gravity of the f's for the various axles, so that the sum of fl for

all the axles is zero. Then the lateral creepage force on all the axles is

$$\left[\frac{1}{v}\frac{dy}{dt} - \theta\right] \times \Sigma(2f) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where $\Sigma(2f)$ is the sum of the creepage coefficients taken over all axles.

The moment due to all axles is

$$\frac{1}{v}\frac{d\theta}{dt} \times \Sigma(2fl^2) + \left(\frac{a^2}{v^2}\cdot\frac{d\theta}{dt}\right) \times \Sigma(2f) + a\lambda y\Sigma\left(\frac{2f}{r}\right) + a\lambda\theta\Sigma\left(\frac{2fl}{r}\right)$$

$$= \frac{1}{v}\frac{d\theta}{dt}\left[\Sigma(2fl^2) + a^2\Sigma(2f)\right] + a\lambda y\Sigma\left(\frac{2f}{r}\right) + a\lambda\theta\Sigma\left(\frac{2fl}{r}\right). \quad . \ (24)$$

Equations (23) and (24) are the basis of much subsequent work on the vibration of trucks.

For slow free motion, for which the forces and moments on the wheelbase are balanced, the force in equation (23) and the moment in equation (24) are both zero. Assume all wheels of equal radius.

From the first, $$\theta = \frac{1}{v}\frac{dy}{dt}$$

and substituting this in the second, and equating to zero,

$$\frac{1}{v^2}\frac{d^2y}{dt^2} + \frac{\left(\frac{a\lambda}{r}\right)\Sigma(2f)}{\Sigma(2fl^2) + a^2\Sigma(2f)}\,y = 0$$

But $$vdt = dx$$

$$\therefore \quad \frac{1}{v^2}\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} + \frac{\left(\frac{a\lambda}{r}\right)\Sigma(2f)}{\Sigma(2fl^2) + a^2\Sigma(2f)}\,y = 0$$

A solution of this is

$$y = y_{\max}\sin\sqrt{\frac{\frac{a\lambda}{r}\,\Sigma(2f)}{\Sigma(2fl^2) + a^2\Sigma(2f)}} \times x$$

and the periodic distance is

$$2\pi\sqrt{\frac{\Sigma(2fl^2) + a^2\Sigma(2f)}{\left(\frac{a\lambda}{r}\right)\Sigma(2f)}}$$

For a 2-axle truck with all axles equally loaded and all wheels the same size, the periodic distance is

$$2\pi\sqrt{\frac{(l^2 + a^2)r}{a\lambda}}$$

For a wheelbase of 8 ft.

$$l = 4 \text{ ft.}$$

$$a = 2.46 \text{ ft.}$$

$$r = 1.5 \text{ ft.}$$

$$\lambda = \tfrac{1}{20}$$

The periodic distance is

$$6.28\sqrt{\frac{(16 + 6.05) \times 1.5}{2.46 \times (\frac{1}{20})}} = 103 \text{ ft.}$$

It will be evident that in general the longer the wheelbase the longer the periodic distance. These formulae are mainly useful as an introduction to the dynamical formulae in the next chapter. In a general way it will be evident that if a truck has approximately a constant periodic distance, then the faster the truck moves, the more quickly will it cover the periodic distance and hence the shorter will be the periodic time of the oscillation. When the periodic time approaches the natural period of rolling of the superstructure on its springs, the conditions will favor the building up of serious oscillations. These oscillations are so much influenced by inertia forces and by flange forces that the theory will be left here until these can be taken into account.

In formulae (23) and (24) above it was assumed that l represented the distance of an axle ahead of the point for which $\Sigma fl = 0$. If some other point is taken from which l is measured, the equations may easily be shown to become:

Lateral force resisting motion is

$$2\Sigma f\left[\frac{dy}{dx} - \theta + l\frac{d\theta}{dx}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Moment resisting rotation is

$$2\Sigma f\left[(l^2 + a^2)\frac{d\theta}{dx} + \frac{\lambda ay}{r} - \left(1 - \frac{\lambda a}{r}\right)l\theta + l\frac{dy}{dx}\right]. \quad . \quad (26)$$

The above equations are in terms of x, the distance moved along the track. They are easily put in terms of time by using the relation $x = vt$, and are:

Lateral force resisting motion is

$$\frac{2\Sigma f}{v}\left[\frac{dy}{dt} - v\theta + l\frac{d\theta}{dt}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Moment resisting rotation is

$$\frac{2\Sigma f}{v}\left[(l^2 + a^2)\frac{d\theta}{dt} + \frac{v\lambda ay}{r} - v\left(1 - \frac{\lambda a}{r}\right)l\theta + l\frac{dy}{dt}\right]. \quad . \quad (28)$$

Chapter 18

MOTION OF TRUCKS ON STRAIGHT TRACK (*Continued*)

The case of a single rigid wheelbase has been studied in the last section. It will proceed down the track oscillating regularly from side to side by an amount depending on the original disturbance which started the oscillation. It should be understood that all the oscillations discussed in this chapter are those which occur at speeds which are low enough to make inertia forces negligible. Let us take a single-axle radius bar guiding truck ahead of a rigid wheelbase, as for example in a 2–6–0 locomotive, shown diagrammatically in Fig. 106. It will

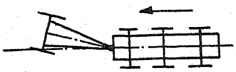


FIG. 106

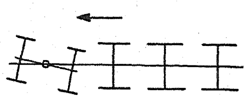


FIG. 107

be evident that if this locomotive moves forward and the guiding truck axis becomes slightly inclined to the axis of the main truck, the guiding truck will swing sideways until its flanges hit the rail. The wheelbase as shown is unstable. If a restraining device is used between the guiding truck and the main truck, to keep the guiding truck in line by means of springs, inclined planes, etc., the combination will still be unstable unless the restraint is sufficient to overcome friction or creepage forces and, therefore, keep all the axles parallel all the time. The combination will then behave on straight track like a 4-axle, 0–8–0 locomotive, except that the leading axle will be able to cushion blows due to rough track. If this locomotive runs backwards—that is, if it is of the 0–6–2 type—it will be evident that the trailing truck will follow the main truck stably and will return to its normal position if accidentally displaced.

Now consider the usual 2-axle pivoted guiding truck in a 4–6–0 locomotive as in Fig. 107. It will be clear that there is nothing to alter

the angle between the guiding and main trucks except the effect of coning which is generally secondary. The guiding truck is neutral and the main truck tends to follow it stably, under conditions depending on the characteristics of the restraint device and on the relative axle loads.

When this locomotive operates backwards, as an 0–6–4, the trailing truck tends to run off sideways and buckle the wheel base, so that the combination tends to be unstable.

This is one reason that such trucks are sometimes fitted with rotational restraints, either in the center plate or in the form of a radius bar, as in Fig. 108. With a combination of high lateral restraint in the center plate and a radius bar, or similar rotational restraint, this truck is stable either leading or trailing and is widely used on electric locomotives which are required to go in either direction. On a single

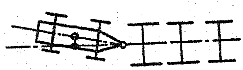


Fig. 108

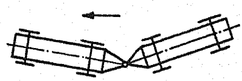


Fig. 109

direction locomotive, such as a 4–6–4 steam engine, it is customary to provide the front truck with a fairly high lateral restraint at the center plate and the rear truck with a radius bar and rather less lateral restraint, thus giving a stable combination for forward motion.

As a further example it may be noted that if two similar trucks are articulated together as in Fig. 109, they will behave just as if they were separate, except that the amount of the oscillation will always be the same for both trucks. This may be seen to be true because, considering one truck of itself, each point of the truck moves along a sine curve and hence the articulation moves along a sine curve. Since the trucks are similar, they will have the same periodic distance and as soon as they have adjusted themselves to steady motion they will continue without any lateral forces being exerted at the articulated joint.

Critical Speed—One Rigid Wheelbase. According to equation (27) the lateral force which resists motion of a truck is

$$\frac{2\Sigma f}{v}\left[\frac{dy}{dt} - v\theta + l\frac{d\theta}{dt}\right].$$

The velocity of lateral motion is

$$\frac{dy}{dt}.$$

Therefore, the rate at which energy is dissipated in lateral motion, which is the product of these two, is

$$\frac{2\Sigma f}{v}\left[\frac{dy}{dt} - v\theta + l\frac{d\theta}{dt}\right]\frac{dy}{dt}.$$

Similarly, from (28) the moment resisting rotation is, omitting terms in λ which arise from coning of the wheels,

$$\frac{2\Sigma f}{v}\left[(l^2 + a^2)\frac{d\theta}{dt} - vl\theta + l\frac{dy}{dt}\right]$$

and hence the rate of dissipation of energy in rotation is

$$\frac{2\Sigma f}{v}\left[(l^2 + a^2)\frac{d\theta}{dt} - vl\theta + l\frac{dy}{dt}\right]\frac{d\theta}{dt}.$$

In this calculation it is sufficiently close to assume that all the motions are represented by sine curves of the same frequency f. If

$$y = Y\cos\omega t$$

$$\theta = \Theta\cos(\omega t + \alpha)$$

$$\omega = 2\pi f.$$

Then substituting, the total rate of loss of energy is

$$\begin{aligned}\frac{2\Sigma f}{v}[Y^2\omega^2\sin^2\omega t &+ v\Theta Y\omega\cos(\omega t + \alpha)\sin\omega t\\ &+ 2l\Theta Y\omega^2\sin(\omega t + \alpha)\sin\omega t\\ &+ (l^2 + a^2)\Theta^2\omega^2\sin^2(\omega t + \alpha)\\ &+ vl\Theta^2\omega\cos(\omega t + \alpha)\sin(\omega t + \alpha)]\end{aligned}$$

The average value of this over a complete period is

$$\frac{2\Sigma f}{v}\left[\frac{Y^2\omega^2}{2} - \frac{v\Theta Y\omega}{2}\sin\alpha + \frac{2l\Theta Y\omega^2}{2}\cos\alpha + \frac{(l^2 + a^2)\Theta^2\omega^2}{2}\right] \quad (29)$$

Now, for convenience, choose the origin of coordinates so that $\Sigma fl = 0$, that is at the center of gravity of the creepage coefficients, then the average rate of loss of energy is

$$\frac{\Sigma f}{v}[Y^2\omega^2 - v\Theta Y\omega \sin\alpha + (l^2 + a^2)\Theta^2\omega^2]$$

It will be evident that if v is small this is always positive. That is, energy is always lost in an oscillation at slow speed. If $\sin\alpha$ is positive, however, which it is for practical cases, there is some value of the speed, v, above which the rate of loss of energy is negative. That is, above this value of speed the truck gains energy during an oscillation. This speed for which there is neither loss nor gain is called the critical speed. To find it we put the loss equal to zero, that is

$$\Sigma f[Y^2\omega^2 - v\Theta Y \sin\alpha + (l^2 + a^2)\Theta^2\omega] = 0$$

or

$$v = \frac{\omega}{\sin\alpha}\left[\frac{Y}{\Theta} + \frac{\Sigma f(l^2 + a^2)}{\Sigma f}\frac{\Theta}{Y}\right]$$

It will be obvious that the larger is $\sin\alpha$, the smaller is v, so that for the lowest value of v, $\sin\alpha = 1$. Then

$$v = \omega\left[\frac{Y}{\Theta} + \frac{\Sigma f(l^2 + a^2)}{\Sigma f}\frac{\Theta}{Y}\right].$$

This is a minimum when

$$\left(\frac{Y}{\Theta}\right)^2 = \frac{\Sigma f(l^2 + a^2)}{\Sigma f}$$

and then, finally

$$V_{\text{critical}} = 2\omega\sqrt{\frac{\Sigma f(l^2 + a^2)}{\Sigma f}}.$$

Critical Speed—Two Wheelbases. We will now consider two trucks articulated together, or a locomotive with a guiding or trailing truck rigidly pivoted at one point. Take the common point as the origin of coordinates and let subscripts $_1$ and $_2$ refer to the two trucks.

Then from (29) the average loss of energy for the two trucks together can be equated to zero, giving

$$\Sigma_1 f[Y^2\omega^2 - v\Theta_1 Y\omega \sin\alpha_1 + 2l\Theta_1 Y\omega^2\cos\alpha_1 + (l^2+a^2)\Theta_1{}^2\omega^2]$$
$$+\Sigma_2 f[Y^2\omega^2 - v\Theta_2 Y\omega \sin\alpha_2 + 2l\Theta_2 Y\omega^2\cos\alpha_2 + (l^2+a^2)\Theta_2{}^2\omega^2] = 0$$

from which

$$\frac{v}{\omega} = \frac{\left\{\begin{aligned}&\Sigma_1 f + 2\Sigma_1 fl\frac{\Theta_1}{Y}\cos\alpha_1 + \Sigma_1 f(l^2+a^2)\left(\frac{\Theta_1}{Y}\right)^2\\ &\quad + \Sigma_2 f + 2\Sigma_2 fl\left(\frac{\Theta_2}{Y}\right)\cos\alpha_2 + \Sigma_2 f(l^2+a^2)\left(\frac{\Theta_2}{Y}\right)^2\end{aligned}\right\}}{\Sigma_1 f\frac{\Theta_1}{Y}\sin\alpha_1 + \Sigma_2 f\frac{\Theta_2}{Y}\sin\alpha_2} \tag{30}$$

This is in the form $X = \dfrac{U}{V}$ and the least value of X is given by $\dfrac{dX}{d\alpha_1} = 0$ and similar equations for α_2, $\dfrac{\Theta_1}{Y}$ and $\dfrac{\Theta_2}{Y}$.

But
$$\frac{dX}{d\alpha_1} = \frac{d}{d\alpha_1}\left(\frac{U}{V}\right) = \frac{V\dfrac{dU}{d\alpha_1} - U\dfrac{dV}{d\alpha_1}}{V^2} = 0$$

Therefore $\dfrac{U}{V} = \dfrac{\left(\dfrac{dU}{d\alpha_1}\right)}{\left(\dfrac{dV}{d\alpha_1}\right)}$ and three similar equations.

Thus
$$\frac{v}{\omega} = -\frac{2\Sigma_1 fl}{\Sigma_1 f}\tan\alpha_1 = -\frac{2\Sigma_2 fl}{\Sigma_2 f}\tan\alpha_2$$

$$\frac{v}{\omega} = \frac{2\Sigma_1 fl\cos\alpha_1 + 2\Sigma_1 f(l^2+a^2)\dfrac{\Theta_1}{Y}}{\Sigma_1 f\sin\alpha_1}$$

$$= \frac{2\Sigma_2 fl\cos\alpha_2 + 2\Sigma_2 f(l^2+a^2)\dfrac{\Theta_2}{Y}}{\Sigma_2 f\sin\alpha_2}$$

$$\therefore \quad \frac{\Theta}{Y}\cos\alpha_1 = \frac{v}{\omega}\frac{\Sigma_1 f}{2\Sigma_1 f(l^2+a^2)} \times \frac{\left(-\dfrac{v}{\omega}\dfrac{\Sigma_1 f}{2\Sigma_1 fl}\right)}{1+\left(\dfrac{v}{\omega}\dfrac{\Sigma_1 f}{2\Sigma_1 fl}\right)^2}$$

$$- \frac{\Sigma_1 fl}{\Sigma_1 f(l^2+a^2)} \times \frac{1}{1+\left(\dfrac{v}{\omega}\dfrac{\Sigma_1 f}{2\Sigma_1 fl}\right)^2}$$

$$\frac{\Theta}{Y}\cos\alpha_1 = -\frac{\Sigma_1 fl}{\Sigma_1 f(l^2+a^2)}$$

Also
$$\frac{\Theta}{Y}\sin\alpha_1 = \frac{\dfrac{v}{\omega}\Sigma_1 f}{2\Sigma_1 f(l^2+a^2)}$$

and
$$\left(\frac{\Theta_1}{Y}\right)^2 \times \Sigma_1 f(l^2+a^2) = \frac{(\Sigma_1 fl)^2+\left(\dfrac{v}{2\omega}\Sigma_1 f\right)^2}{\Sigma_1 f(l^2+a^2)}$$

Substituting in (30) above

$$\frac{1}{2}\left(\frac{v}{\omega}\right)^2\left[\frac{(\Sigma_1 f)^2}{\Sigma_1 f(l^2+a^2)}+\frac{(\Sigma_2 f)^2}{\Sigma_2 f(l^2+a^2)}\right] = \Sigma_1 f - \frac{(\Sigma_1 fl)^2}{\Sigma_1 f(l^2+a^2)}$$

$$+\frac{\left(\dfrac{v}{2\omega}\Sigma_1 f\right)^2}{\Sigma_1 f(l^2+a^2)} + \Sigma_2 f - \frac{(\Sigma_2 fl)^2}{\Sigma_2 f(l^2+a^2)} + \frac{\left(\dfrac{v}{2\omega}\Sigma_2 f\right)^2}{\Sigma_2 f(l^2+a^2)}$$

$$\therefore \left(\frac{v}{2\omega}\right)^2$$

$$= \frac{\dfrac{(\Sigma_1 f)[\Sigma_1 f(l^2+a^2)]-(\Sigma_1 fl)^2}{\Sigma_1 f(l^2+a^2)} + \dfrac{(\Sigma_2 f)[\Sigma_2 f(l^2+a^2)]-(\Sigma_2 fl)^2}{\Sigma_2 f(l^2+a^2)}}{\dfrac{(\Sigma_1 f)^2}{\Sigma_1 f(l^2+a^2)}+\dfrac{(\Sigma_2 f)^2}{\Sigma_2 f(l^2+a^2)}} \qquad (31)$$

In many practical cases this formula may be simplified; for example, if the two wheelbases are symmetrical about the common point

each (l) in one becomes ($-l$) in the other. Then the critical speed is the same as for one wheelbase alone, if the frequency is unchanged, and

$$\left(\frac{v}{2\omega}\right)^2 = \frac{(\Sigma_1 f)[\Sigma_1 f(l^2 + a^2)] - (\Sigma_1 fl)^2}{(\Sigma_1 f)^2} \quad . \quad . \quad (32)$$

If the common point is such that $\Sigma_2 fl = 0$, for example if truck 2 has a center plate which supports part of truck 1, the second expression in the numerator of (31) becomes simply $\Sigma_2 f$.

It is easy to see that if the direction of motion of the trucks is changed this is the same as reversing the signs of all the l's and the formulae show that this merely changes the signs of $\cos \alpha_1$, and $\cos \alpha_2$. Take for example a 4–6–0 locomotive with a high resistance truck, so that the truck center plate is the common point. Let the 6-wheel driving wheelbase be 1 and the 4-wheel truck be 2.

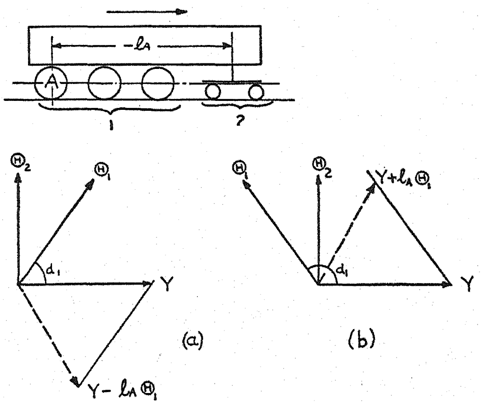


FIG. 110

In Fig. 110 (a) is a vector diagram for the forward direction of motion. The motion of the rear wheel, A, is represented by the vector $Y - l_A\Theta_1$, shown by a dotted line. In Fig. 110 (b) is the corresponding diagram for motion in the opposite direction. The l's are reversed, reversing the cosine of α_1 but not changing the length of the vector which represents the motion of A, which is now $Y + l_A\Theta_1$. This has

a most important application to symmetrical articulated locomotives like a $4-6+6-4$ type. At the critical speed the two halves move with the same amplitude regardless of the presence or absence of the articulated joint and, therefore, the critical speed of such a locomotive is the critical speed of one half, as long as the joint is not otherwise restrained.

Critical Speed—Three Rigid Wheelbases. In order to cover the common type of locomotive which has a rigid frame for the drivers and pivoted leading and trailing trucks we must consider the combination of 3 rigid wheelbases with 2 common points. Let us take as origin a point on the middle wheelbase such that, for this wheelbase, $\Sigma fl = 0$, that is, at the center of gravity of the creepage coefficients. Let this middle wheelbase be referred to by subscript $_0$ and the other two by subscripts $_1$ and $_2$. For wheelbases $_1$ and $_2$ let the common points with wheelbase $_0$ be distances d_1 and d_2 ahead of the origin and let the l's for these two wheelbases be measured from the common points. From (29) and putting

$$Y_1^2 = Y_0^2 + (d_1\Theta_0)^2 + 2Y_0 d_1 \Theta_0 \cos\alpha_0$$

$$Y_2^2 = Y_0^2 + (d_2\Theta_0)^2 + 2Y_0 d_2 \Theta_0 \cos\alpha_0$$

we have the condition for critical speed,

$$\Sigma_0 f\left[Y_0^2 - \frac{v}{\omega}\Theta_0 Y_0 \sin\alpha_0 + (l^2 + a^2)\Theta_0^2\right]$$

$$+\Sigma_1 f\left[Y_0^2 + d_1^2\Theta_0^2 + 2Y_0 d_1\Theta_0\cos\alpha_0 + \left(2l\cos\alpha_1 - \frac{v}{\omega}\sin\alpha_1\right)\Theta_1\right.$$

$$\left.\sqrt{Y_0^2 + d_1^2\Theta_0^2 + 2Y_0 d_1\Theta_0\cos\alpha_0} + (l^2 + a^2)\Theta_1^2\right]$$

$$+\Sigma_2 f\left[Y_0^2 + d_2^2\Theta_0^2 + 2Y_0 d_2\Theta_0\cos\alpha_0 + \left(2l\cos\alpha_2 - \frac{v}{\omega}\sin\alpha_2\right)\Theta_2\right.$$

$$\left.\sqrt{Y_0^2 + d_2^2\Theta_0^2 + 2Y_0 d_2\Theta_0\cos\alpha_0} + (l^2 + a^2)\Theta_2^2\right] = 0$$

$$\therefore\ \frac{v}{\omega}=\frac{\left\{\begin{array}{l}Y_0^2[\Sigma_0 f+\Sigma_1 f+\Sigma_2 f]+\Theta_0^2[\Sigma_0 f(l^2+a^2)+\Sigma_1 f d_1^2\\ \quad +\Sigma_2 f d_2^2]+\Theta_1^2\Sigma_1 f(l^2+a^2)+\Theta_2^2\Sigma_2 f(l^2+a^2)\\ \quad +2d_1\cos\alpha_0 Y_0\Theta_0\Sigma_1 f+2d_2\cos\alpha_0 Y_0\Theta_0\Sigma_2 f\\ \quad +2\Sigma_1 fl\cos\alpha_1\Theta_1\sqrt{Y_0^2+d_1^2\Theta_0^2+2Y_0 d_1\Theta_0\cos\alpha_0}\\ \quad +2\Sigma_2 fl\cos\alpha_2\Theta_2\sqrt{Y_0^2+d_2^2\Theta_0^2+2Y_0 d_2\Theta_0\cos\alpha_0}\end{array}\right\}}{\left\{\begin{array}{l}\Sigma_0 f\Theta_0 Y_0\sin\alpha_0\\ \quad +\Sigma_1 f\sin\alpha_1\Theta_1\sqrt{Y_0^2+d_1^2\Theta_0^2+2Y_0 d_1\Theta_0\cos\alpha_0}\\ \quad +\Sigma_2 f\sin\alpha_2\theta_2\sqrt{Y_0^2+d_2^2\Theta_0^2+2Y_0 d_2\Theta_0\cos\alpha_0}\end{array}\right\}} \tag{33}$$

As in the previous section we write the conditions that $\frac{\Theta_0}{Y_0}$, $\frac{\Theta_1}{Y_0}$, $\frac{\Theta_2}{Y_0}$, α_0, α_1 and α_2 have the values which will give the lowest value of the critical speed. These are:

$$\Theta_0 \quad \frac{v}{\omega}=\frac{\left\{\begin{array}{l}2\frac{\Theta_0}{Y_0}[\Sigma_0 f(l^2+a^2)+\Sigma_1 f d_1^2+\Sigma_2 f d_2^2]+2d_1\cos\alpha_0\Sigma_1 f\\ \quad +2d_2\cos\alpha_0\Sigma_2 f+2\Sigma_1 fl\cos\alpha_1\frac{\Theta_1}{Y_0}\left(\dfrac{2d_1^2\frac{\Theta_0}{Y_0}+2d_1\cos\alpha_0}{2Y_1/Y_0}\right)\\ \quad +2\Sigma_2 fl\cos\alpha_2\frac{\Theta_2}{Y_0}\left(\dfrac{2d_2^2\frac{\Theta_0}{Y_0}+2d_2\cos\alpha_0}{2Y_2/Y_0}\right)\end{array}\right\}}{\left\{\begin{array}{l}\Sigma_0 f\sin\alpha_0+\Sigma_1 f\sin\alpha_1\frac{\Theta_1}{Y_0}\left(\dfrac{2d_1^2\frac{\Theta_0}{Y_0}+2d_1\cos\alpha_0}{2Y_1/Y_0}\right)\\ \quad +\Sigma_2 f\sin\alpha_2\frac{\Theta_2}{Y_0}\left(\dfrac{2d_2^2\frac{\theta_0}{Y_0}+2d_2\cos\alpha_0}{2Y_2/Y_0}\right)\end{array}\right\}} \tag{34}$$

$$\Theta_1 \quad \frac{v}{\omega} = \frac{2\dfrac{\Theta_1}{Y_0}\Sigma_1 f(l^2+a^2) + 2\Sigma_1 fl\cos\alpha_1\dfrac{Y_1}{Y_0}}{\Sigma_1 f\sin\alpha_1\dfrac{Y_1}{Y_0}} \quad \ldots\ldots \quad (35)$$

$$\Theta_2 \quad \frac{v}{\omega} = \frac{2\dfrac{\Theta_2}{Y_0}\Sigma_2 f(l^2+a^2) + 2\Sigma_2 fl\cos\alpha_2\dfrac{Y_2}{Y_0}}{\Sigma_2 f\sin\alpha_2\dfrac{Y_2}{Y_0}} \quad \ldots\ldots \quad (36)$$

$$\alpha_0 \quad \frac{v}{\omega} = \frac{\left\{\begin{array}{l} -2d_1\dfrac{\Theta_0}{Y_0}\Sigma_1 f\sin\alpha_0 - 2d_2\dfrac{\Theta_0}{Y_0}\Sigma_2 f\sin\alpha_0 - 2\Sigma_1 fl \\ \quad \cos\alpha_1\dfrac{\Theta_1}{Y_0}\dfrac{d_1\dfrac{\Theta_0}{Y_0}\sin\alpha_0}{Y_1/Y_0} - 2\Sigma_2 fl\cos\alpha_2\dfrac{\Theta_2}{Y_0}\dfrac{d_2\dfrac{\Theta_0}{Y_0}\sin\alpha_0}{Y_2/Y_0}\end{array}\right\}}{\left\{\begin{array}{l} \Sigma_0 f\dfrac{\Theta_0}{Y_0}\cos\alpha_0 - \Sigma_1 f\sin\alpha_1\dfrac{\Theta_1}{Y_0}\dfrac{d_1\dfrac{\Theta_0}{Y_0}\sin\alpha_0}{Y_1/Y_0} \\ \quad -\Sigma_2 f\sin\alpha_2\dfrac{\Theta_2}{Y_0}\dfrac{d_2\dfrac{\Theta_0}{Y_0}\sin\alpha_0}{Y_2/Y_0}\end{array}\right\}} \quad (37)$$

$$\alpha_1 \quad \frac{v}{\omega} = \frac{-2\Sigma_1 fl\sin\alpha_1\dfrac{\Theta_1}{Y_0}\dfrac{Y_1}{Y_0}}{\Sigma_1 f\cos\alpha_1\dfrac{\Theta_1}{Y_0}\dfrac{Y_1}{Y_0}} = \frac{-2\Sigma_1 fl}{\Theta_1 f}\tan\alpha_1 \quad \ldots\ldots \quad (38)$$

$$\alpha_2 \quad \frac{v}{\omega} = -\frac{2\Sigma_2 fl\tan\alpha_2}{\Sigma_2 f} \quad \ldots\ldots\ldots\ldots \quad (39)$$

Hence

$$\frac{\Theta_1}{Y_0}\cos\alpha_1 = \frac{\dfrac{v}{2\omega}\dfrac{\Sigma_1 f}{\Sigma_1 f(l^2+a^2)}\dfrac{Y_1}{Y_0}\left(-\dfrac{v}{2\omega}\dfrac{\Sigma_1 f}{\Sigma_1 fl}\right) - \dfrac{\Sigma_1 fl}{\Sigma_1 f(l^2+a^2)} \times \dfrac{Y_1}{Y_0}}{1+\left(\dfrac{v}{2\omega}\dfrac{\Sigma_1 f}{\Sigma_1 fl}\right)^2}$$

$$= -\frac{Y_1}{Y_0}\frac{\Sigma_1 fl}{\Sigma_1 f(l^2+a^2)}$$

$$\frac{\Theta_1}{Y_0}\sin\alpha_1 = \frac{Y_1}{Y_0}\frac{v}{2\omega}\frac{\Sigma_1 f}{\Sigma_1 f(l^2+a^2)}$$

$$\left(\frac{\Theta_1}{Y_0}\right)^2 \Sigma_1 f(l^2+a^2) = \left(\frac{Y_1}{Y_0}\right)^2\left[\frac{(\Sigma_1 fl)^2 + \left(\dfrac{v}{2\omega}\Sigma_1 f\right)^2}{\Sigma_1 f(l^2+a^2)}\right]$$

Substituting in (37) we obtain:

$$\frac{v}{2\omega} = -\frac{d_1\Sigma_1 f + d_2\Sigma_2 f - d_1\dfrac{(\Sigma_1 fl)^2}{\Sigma_1 f(l^2+a^2)} - d_2\dfrac{(\Sigma_2 fl)^2}{\Sigma_2 f(l^2+a^2)}}{\Sigma_0 f - \dfrac{v}{2\omega}\tan\alpha_0\left[\dfrac{d_1(\Sigma_1 f)^2}{\Sigma_1 f(l^2+a^2)} + \dfrac{d_2(\Sigma_2 f)^2}{\Sigma_2 f(l^2+a^2)}\right]}\tan\alpha_0$$

$$\therefore \quad \tan\alpha_0 = \frac{\Sigma_0 f\dfrac{v}{2\omega}}{\left\{\begin{aligned}&\frac{d_1}{\Sigma_1 f(l^2+a^2)}\Big[(\Sigma_1 fl)^2 - \Sigma_1 f\Sigma_1 f(l^2+a^2)\\ &\quad + \left(\frac{v}{2\omega}\right)^2(\Sigma_1 f)^2\Big] + \frac{d_2}{\Sigma_2 f(l^2+a^2)}\Big[(\Sigma_2 fl)^2\\ &\quad - \Sigma_2 f\Sigma_2 f(l^2+a^2) + \left(\frac{v}{2\omega}\right)^2(\Sigma_2 f)^2\Big]\end{aligned}\right\}}$$

Again, substituting in (34):

$$\frac{v}{2\omega}=\frac{\left\{\begin{aligned}&\frac{\Theta_0}{Y_0}\left[\Sigma_0 f(l^2+a^2)+\Sigma_1 f d_1{}^2+\Sigma_2 f d_2{}^2-\frac{d_1{}^2(\Sigma_1 fl)^2}{\Sigma_1 f(l^2+a^2)}\right.\\&\left.-\frac{d_2{}^2(\Sigma_2 fl)^2}{\Sigma_2 f(l^2+a^2)}\right]+\cos\alpha_0\left[d_1\Sigma_1 f+d_2\Sigma_2 f\right.\\&\left.-\frac{d_1(\Sigma_1 fl)^2}{\Sigma_1 f(l^2+a^2)}-\frac{d_2(\Sigma_2 fl)^2}{\Sigma_2 f(l^2+a^2)}\right]\end{aligned}\right\}}{\left\{\begin{aligned}&\Sigma_0 f\sin\alpha_0+\frac{\Theta_0}{Y_0}\frac{v}{2\omega}\left[\frac{d_1{}^2(\Sigma_1 f)^2}{\Sigma_1 f(l^2+a^2)}+\frac{d_2{}^2(\Sigma_2 f)^2}{\Sigma_2 f(l^2+a^2)}\right]\\&+\cos\alpha_0\frac{v}{2\omega}\left[\frac{d_1(\Sigma_1 f)^2}{\Sigma_1 f(l^2+a^2)}+\frac{d_2(\Sigma_2 f)^2}{\Sigma_2 f(l^2+a^2)}\right]\end{aligned}\right\}}$$

$$\therefore\ \frac{\Theta_0}{Y_0\cos\alpha_0}=\frac{\left\{\begin{aligned}&\left(\Sigma_0 f\frac{v}{2\omega}\right)^2+\left\{\frac{d_1}{\Sigma_1 f\,(l^2+a^2)}\left[(\Sigma_1 fl)^2\right.\right.\\&\left.-\Sigma_1 f\,\Sigma_1 f\,(l^2+a^2)+\left(\frac{v}{2\omega}\right)^2(\Sigma_1 f)^2\right]\\&+\frac{d_2}{\Sigma_2 f\,(l^2+a^2)}\left[(\Sigma_2 fl)^2-\Sigma_2 f\,\Sigma_2 f(l^2+a^2)\right.\\&\left.\left.+\left(\frac{v}{2\omega}\right)^2(\Sigma_2 f)^2\right]\right\}^2\end{aligned}\right\}}{\left\{\begin{aligned}&\left\{\frac{d_1}{\Sigma_1 f\,(l^2+a^2)}\left[(\Sigma_1 fl)^2-\Sigma_1 f\,\Sigma_1 f(l^2+a^2)\right.\right.\\&\left.+\left(\frac{v}{2\omega}\right)^2(\Sigma_1 f)^2\right]+\frac{d_2}{\Sigma_2 f(l^2+a^2)}\left[(\Sigma_2 fl)^2\right.\\&\left.\left.-\Sigma_2 f\,\Sigma_2 f(l^2+a^2)+\left(\frac{v}{2\omega}\right)^2(\Sigma_2 f)^2\right]\right\}\left\{\Sigma_0 f\,(l^2+a^2)\right.\\&-\frac{d_1{}^2}{\Sigma_1 f(l^2+a^2)}\left[(\Sigma_1 fl)^2-\Sigma_1 f\,\Sigma_1 f(l^2+a^2)\right.\\&\left.+\left(\frac{v}{2\omega}\right)^2(\Sigma_1 f)^2\right]-\frac{d_2{}^2}{\Sigma_2 f(l^2+a^2)}\left[(\Sigma_2 fl)^2\right.\\&\left.\left.-\Sigma_2 f\,\Sigma_2 f(l^2+a^2)+\left(\frac{v}{2\omega}\right)^2(\Sigma_2 f)^2\right]\right\}\end{aligned}\right\}}$$

$$\therefore \quad \frac{\Theta_0}{Y_0}\cos\alpha_0 = \frac{\left\{\begin{array}{l} \dfrac{d_1}{\Sigma_1 f\,(l^2+a^2)}\left[(\Sigma_1 fl)^2 - \Sigma_1 f\,\Sigma_1 f(l^2+a^2) \right. \\ \quad \left. + \left(\dfrac{v}{2\omega}\right)^2(\Sigma_1 f)^2\right] + \dfrac{d_2}{\Sigma_2 f\,(l^2+a^2)}\left[(\Sigma_2 fl)^2 \right. \\ \quad \left. - \Sigma_2 f\,\Sigma_2 f(l^2+a^2) + \left(\dfrac{v}{2\omega}\right)^2(\Sigma_2 f)^2\right] \end{array}\right\}}{\left\{\begin{array}{l} \Sigma_0 f(l^2+a^2) - \dfrac{{d_1}^2}{\Sigma_1 f\,(l^2+a^2)}\left[(\Sigma_1 fl)^2 \right. \\ \quad \left. - \Sigma_1 f\,\Sigma_1 f(l^2+a^2) + \left(\dfrac{v}{2\omega}\right)^2(\Sigma_1 f)^2\right] \\ \quad - \dfrac{{d_2}^2}{\Sigma_2 f\,(l^2+a^2)}\left[(\Sigma_2 fl)^2 - \Sigma_2 f\,\Sigma_2 f(l^2+a^2) \right. \\ \quad \left. + \left(\dfrac{v}{2\omega}\right)^2(\Sigma_2 f)^2\right] \end{array}\right\}}$$

$$\therefore \frac{\Theta_0}{Y_0}\sin\alpha_0 = \frac{\Sigma_0 f\,\dfrac{v}{2\omega}}{\left\{\begin{array}{l} \Sigma_0 f(l^2+a^2) - \dfrac{{d_1}^2}{\Sigma_1 f(l^2+a^2)}\left[(\Sigma_1 fl)^2 \right. \\ \quad \left. - \Sigma_1 f\,\Sigma_1 f(l^2+a^2) + \left(\dfrac{v}{2\omega}\right)^2(\Sigma_1 f)^2\right] \\ \quad - \dfrac{{d_2}^2}{\Sigma_2 f(l^2+a^2)}\left[(\Sigma_2 fl)^2 - \Sigma_2 f\Sigma_2 f(l^2+a^2) \right. \\ \quad \left. + \left(\dfrac{v}{2\omega}\right)^2(\Sigma_2 f)^2\right] \end{array}\right\}}$$

Now substitute in (33):

$$\Sigma_0 f + \Sigma_1 f + \Sigma_2 f + \left(\frac{\Theta_0}{Y_0}\right)^2[\Sigma_0 f\,(l^2+a^2) + {d_1}^2\Sigma_1 f + {d_2}^2\Sigma_2 f]$$

$$+\left(\frac{Y_1}{Y_0}\right)^2\left[\frac{(\Sigma_1 fl)^2 + \left(\dfrac{v}{2\omega}\,\Sigma_1 f\right)^2}{\Sigma_1 f(l^2+a^2)}\right] + \left(\frac{Y_2}{Y_0}\right)^2\left[\frac{(\Sigma_2 fl)^2 + \left(\dfrac{v}{2\omega}\,\Sigma_2 f\right)^2}{\Sigma_2 f(l^2+a^2)}\right]$$

$$+ 2[d_1\Sigma_1 f + d_2\Sigma_2 f]\frac{\Theta_0}{Y_0}\cos\alpha_0 - 2\left(\frac{Y_1}{Y_0}\right)^2\frac{(\Sigma_1 fl)^2}{\Sigma_1 f(l^2+a^2)}$$

$$- 2\left(\frac{Y_2}{Y_0}\right)^2\frac{(\Sigma_2 fl)^2}{\Sigma_2 f(l^2+a^2)} - 2\Sigma_0 f\frac{v}{2\omega}\frac{\Theta_0}{Y_0}\sin\alpha_0$$

$$- 2\left(\frac{Y_1}{Y_0}\right)^2\left(\frac{v}{2\omega}\right)^2\frac{(\Sigma_1 f)^2}{\Sigma_1 f(l^2+a^2)} - 2\left(\frac{Y_2}{Y_0}\right)^2\left(\frac{v}{2\omega}\right)^2\frac{(\Sigma_2 f)^2}{\Sigma_2 f(l^2+a^2)} = 0$$

$$\therefore\ \Sigma_0 f + \Sigma_1 f + \Sigma_2 f + \left(\frac{\Theta_0}{Y_0}\right)^2[\Sigma_0 f(l^2+a^2) + d_1{}^2\Sigma_2 f + d_2{}^2\Sigma_2 f]$$

$$+ 2[d_1\Sigma_1 f + d_2\Sigma_2 f]\frac{\Theta_0}{Y_0}\cos\alpha_0 - 2\Sigma_0 f\frac{v}{2\omega}\frac{\Theta_0}{Y_0}\sin\alpha_0$$

$$-\left[1 + d_1{}^2\left(\frac{\Theta_0}{Y_0}\right)^2 + 2d_1\frac{\Theta_0}{Y_0}\cos\alpha_0\right]\left[\frac{(\Sigma_1 fl)^2 + \left(\frac{v}{2\omega}\Sigma_1 f\right)^2}{\Sigma_1 f(l^2+a^2)}\right]$$

$$-\left[1 + d_2{}^2\left(\frac{\Theta_0}{Y_0}\right)^2 + 2d_2\frac{\Theta_0}{Y_0}\cos\alpha_0\right]\left[\frac{(\Sigma_2 fl)^2 + \left(\frac{v}{2\omega}\Sigma_2 f\right)^2}{\Sigma_2 f(l^2+a^2)}\right] = 0$$

Finally, substituting for $\frac{\Theta_0}{Y_0}$ and α_0 in this equation, we obtain:

$$\left(\frac{v}{2\omega}\right)^4\left[\frac{(\Sigma_1 f)^2(\Sigma_2 f)^2(d_1-d_2)^2}{\Sigma_1 f(l^2+a^2)\Sigma_2 f(l^2+a^2)}\right]$$

$$+\left(\frac{v}{2\omega}\right)^2\left[\frac{(d_1-d_2)^2}{\Sigma_1 f(l^2+a^2)\Sigma_2 f(l^2+a^2)}\right]$$

$$[(\Sigma_1 f)^2\{(\Sigma_2 fl)^2 - \Sigma_2 f\Sigma_2 f(l^2+a^2)\}$$

$$+ (\Sigma_2 f)^2\{(\Sigma_1 fl)^2 - \Sigma_1 f\Sigma_1 f(l^2+a^2)\}]$$

$$-\left(\frac{v}{2\omega}\right)^2\left[(\Sigma_0 f)^2 + \frac{(\Sigma_1 f)^2}{\Sigma_1 f(l^2+a^2)}\{\Sigma_0 f(l^2+a^2) + d_1{}^2\Sigma_0 f\}\right.$$

$$\left.+ \frac{(\Sigma_2 f)^2}{\Sigma_2 f(l^2+a^2)}\{\Sigma_0 f(l^2+a^2) + d_2{}^2\,\Sigma_0 f\}\right]$$

$$+\frac{(d_1-d_2)^2}{\Sigma_1 f(l^2+a^2)\Sigma_2 f(l^2+a^2)}[(\Sigma_1 fl)^2-\Sigma_1 f\Sigma_1 f(l^2+a^2)]$$

$$[(\Sigma_2 fl)^2-\Sigma_2 f\Sigma_2 f(l^2+a^2)]+\Sigma_0 f\Sigma_0 f(l^2+a^2)$$

$$-\left[\frac{(\Sigma_1 fl)^2-\Sigma_1 f\Sigma_1 f(l^2+a^2)}{\Sigma_1 f(l^2+a^2)}\right][\Sigma_0 f(l^2+a^2)+d_1{}^2\Sigma_0 f]$$

$$-\left[\frac{(\Sigma_2 fl)^2-\Sigma_2 f\Sigma_2 f(l^2+a^2)}{\Sigma_2 f(l^2+a^2)}\right][\Sigma_0 f(l^2+a^2)+d_2{}^2\Sigma_0 f]=0$$

Let $\dfrac{-(\Sigma_1 fl)^2+\Sigma_1 f\Sigma_1 f(l^2+a^2)}{\Sigma_1 f(l^2+a^2)}=D_1 \quad \dfrac{(\Sigma_1 f)^2}{\Sigma_1 f(l^2+a^2)}=E_1$

$$\frac{-(\Sigma_2 fl)^2+\Sigma_2 f\Sigma_2 f(l^2+a^2)}{\Sigma_2 f(l^2+a^2)}=D_2 \quad \frac{(\Sigma_2 f)^2}{\Sigma_2 f(l^2+a^2)}=E_2$$

$$\Sigma_0 f(l^2+a^2)+d_1{}^2\Sigma_0 f=F_1 \quad \Sigma_0 f(l^2+a^2)+d_2{}^2\Sigma_0 f=F_2$$

Then

$$\left(\frac{v}{2\omega}\right)^4 E_1E_2(d_1-d_2)^2$$

$$-\left(\frac{v}{2\omega}\right)^2[(E_1D_2+E_2D_1)(d_1-d_2)^2+(\Sigma_0 f)^2+E_1F_1+E_2F_2]$$

$$+D_1D_2(d_1-d_2)^2+\Sigma_0 f\Sigma_0 f(l^2+a^2)+D_1F_1+D_2F_2=0\ldots(40)$$

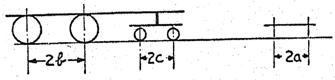


FIG. 111

Particular Cases of Critical Speed. (1) Locomotive, type 4–4–0 (Fig. 111).

Let $2a$ = distance between rails (wheel contacts)
$2b$ = rigid driving wheelbase
$2c$ = truck wheelbase
d = distance between centers of wheelbases
f_1 = creepage coefficient for driving wheels
f_2 = creepage coefficient for truck wheels.

Then, referring to equation (31)

$$\Sigma_1 f = 2f_1$$

$$\Sigma_1 fl = 2f_1 d$$

$$\Sigma_1 f\,(l^2 + a^2 = 2f_1(d^2 + b^2 + a^2)$$

$$\Sigma_2 f = 2f_2$$

$$\Sigma_2 fl = 0$$

$$\Sigma_2 f\,(l^2 + a^2) = 2f_2(c^2 + a^2)$$

Therefore

$$\left(\frac{v}{2\omega}\right)^2 = \frac{f_1\left[1 - \dfrac{d^2}{d^2 + b^2 + a^2}\right] + f_2}{\left(\dfrac{f_1}{d^2 + b^2 + a^2}\right) + \left(\dfrac{f_2}{c^2 + a^2}\right)}.$$

It will easily be seen that in most cases the pivoted truck has a great effect and the critical speed of the combination is not much different from that of the truck alone.

For the truck,

$$\left(\frac{v}{2\omega}\right)^2 = c^2 + a^2.$$

For the combination,

$$\left(\frac{v}{2\omega}\right)^2 = (c^2 + a^2)\left[\frac{f_1(b^2 + a^2) + f_2(d^2 + b^2 + a^2)}{f_1(c^2 + a^2) + f_2(d^2 + b^2 + a^2)}\right]$$

It should be remembered that this applies only when the whole combination oscillates with a single frequency.

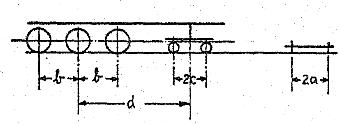


FIG. 112

(2) Locomotive, type 4–6–0 (Fig. 112).

Here

$$\Sigma_1 f = 3f_1$$

$$\Sigma_1 fl = 3f_1 d$$

$$\Sigma_1 f\,(l^2 + a^2) = f_1(3d^2 + 2b^2 + 3a^2)$$

$$\Sigma_2 f = 2f_2$$

$$\Sigma_2 fl = 0$$

$$\Sigma_2 f\,(l^2 + a^2) = 2f_2(c^2 + a^2)$$

Therefore

$$\left(\frac{v}{2\omega}\right)^2 = \frac{3f_1 - \dfrac{9f_1 d^2}{3d^2 + 2b^2 + 3a^2} + 2f_2}{\dfrac{9f_1}{3d^2 + 2b^2 + 3a^2} + \dfrac{2f_2}{c^2 + a^2}}$$

$$= (c^2 + a^2)\left[\frac{f_1(\frac{2}{3}b^2 + a^2) + \frac{2}{3}f_2(d^2 + \frac{2}{3}b^2 + a^2)}{f_1(c^2 + a^2) + \frac{2}{3}f_2(d^2 + \frac{2}{3}b^2 + a^2)}\right]$$

This is a slightly less critical speed than for the 4–4–0 engine with the same wheelbase.

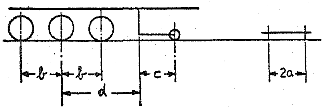


FIG. 113.

(3) Locomotive, type 2–6–0 (Fig. 113).

For the radius bar truck,

$$\Sigma_2 f = f_2$$

$$\Sigma_2 fl = -f_2 c$$

$$\Sigma_2 f\,(l^2 + a^2) = f_2(c^2 + a^2)$$

Therefore

$$\left(\frac{v}{2\omega}\right)^2 = \frac{3f_1 - \dfrac{9f_1 d^2}{3d^2 + 2b^2 + 3a^2} + f_2 - f_2\dfrac{c^2}{c^2 + a^2}}{\dfrac{9f_1}{3d^2 + 2b^2 + 3a^2} + \dfrac{f_2}{c^2 + a^2}}$$

$$= a^2\left[\frac{f_1(\frac{2}{3}b^2 + a^2) + \frac{1}{3}f_2(d^2 + \frac{2}{3}b^2 + a^2)\dfrac{a^2}{a^2 + c^2}}{f_1 a^2 + \frac{1}{3}f_2\left((d^2 + \frac{2}{3}b^2) + a^2\right)\left(\dfrac{a^2}{a^2 + c^2}\right)}\right]$$

(4) Locomotive, type 4–8–2 (Fig. 114).

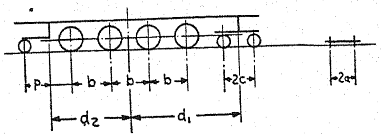


FIG. 114

Let $2a$ = distance between rails
$3b$ = rigid driving wheelbase
$2c$ = leading truck wheelbase
p = length of trailing truck radius bar
d_1 = distance of leading truck center plate ahead of center of driving wheelbase
d_2 = distance of trailing truck pivot behind center of driving wheelbase
f_0 = creepage coefficient for drivers
f_1 = creepage coefficient for leading truck
f_2 = creepage coefficient for trailing truck

Then in formula (40)

$$\Sigma_0 f = 4f_0$$

$$\Sigma_0 f\,(l^2 + a^2) = f_0\,(5b^2 + 4a^2)$$

$$D_1 = 2f_1 \qquad E_1 = \frac{2f_1}{c^2 + a^2}$$

$$D_2 = f_2\frac{a^2}{p^2 + a^2} \qquad E_2 = \frac{f_2}{p^2 + a^2}$$

$$F_1 = (5b^2 + 4a^2 + 4d_1{}^2)f_0$$

$$F_2 = (5b^2 + 4a^2 + 4d_2{}^2)f_0$$

Then
$$\left(\frac{v}{2\omega}\right)^4 \times \frac{2f_1f_2(d_1 + d_2)^2}{(c^2 + a^2)\,(p^2 + a^2)}$$

$$-\left(\frac{v}{2\omega}\right)^2 \left[\frac{2f_1f_2(c^2 + 2a^2)\,(d_1 + d_2)^2}{(p^2 + a^2)\,(c^2 + a^2)}\right.$$

$$+ 16f_0^2 + \frac{2f_1f_0}{(c^2+a^2)}(5b^2+4a^2+4d_1^2) + \frac{f_2f_0}{(p^2+a^2)}(5b^2+4a^2+4d_2^2)\Bigg]$$

$$+\left[\frac{2f_1f_2a^2}{p^2+a^2}(d_1+d_2)^2+4f_0^2(5b^2+4a^2)+2f_1f_0(5b^2+4a^2+4d_1^2)\right.$$

$$\left.+\frac{f_2f_0a^2}{p^2+a^2}(5b^2+4a^2+4d_2^2)\right] = 0$$

which has its lowest root in the neighborhood of

$$v = 2\omega a$$

Chapter 19

MOTION OF TRUCKS ON CURVES

First let us consider a simple 2-axle truck with equal loading on all wheels and no coning of the wheel contours.

The leading axle will run with the outer wheel flange against the outer rail and in general the rear axle will run nearly tangentially. (See Fig. 115.) The center of rotation of the truck is then close to 0, the center of the rear axle. The center of rotation must be on the

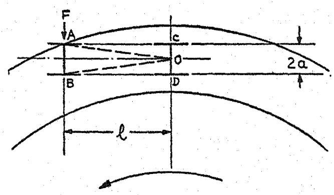


Fig. 115

radius which is at right angles to the truck center line and must also be chosen so that the slip or creep forces do not give any resultant along the center line if the truck is to move freely. 0 satisfies these conditions. Then, if the coefficient of friction is taken as φ, the frictional moment about 0 required to turn the truck is

$$\varphi W(a + \sqrt{l^2 + a^2}$$

where W = the weight per axle
and since this moment is produced by a flange force F at the front axle, at a distance l,

$$F = \varphi W\left(\frac{a + \sqrt{l^2 + a^2}}{l}\right)$$

A more elaborate calculation will show that the center of rotation

is actually slightly behind the rear axle, but the difference is not worth the extra calculation.

In practice it is unnecessary to keep the square root, which is inconvenient to use in general discussions and it is quite accurate enough to use the approximation

$$F = \frac{\varphi W(a + l)}{l}$$

This may be looked upon as due to *turning* of the axle whose center acts as center of rotation and *sliding sideways* of the front axle.

Then in order that the truck may be in equilibrium, there will be a force $\frac{\varphi Wa}{l}$ acting on the rear axle, which is, of course, not sufficient to slide it sideways. (See Fig. 116.)

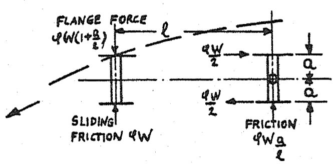


FIG. 116

In addition to the friction and flange forces there may be others due to centrifugal force and superelevation of the track. These both act laterally at the center of mass.

The centrifugal force is $\frac{Mv^2}{R}$ acting outwards.

where M is the mass involved
v is the speed
R is the radius of curvature
g is the acceleration due to gravity

The superelevation of the outer rail produces an inward force $\frac{Mgh}{G}$

where h is the elevation of the outer rail
G is the track gage

Thus the net outward force, acting at the center of mass is

$$\frac{Mv^2}{R} - \frac{Mgh}{G}$$

A common rule limits the comfortable speed on a curve to that for which the elevation is 3 in. less than would be required for equilibrium. This is equivalent to a net outward force amounting to 5.3% of the weight and this figure may be used unless the particular conditions are known. Applying this to the 2-axle truck with a center plate in the middle, the weight will be $2W$ and the net outward force will be

$$0.053 \times 2W$$

which must be resisted at the two axles as shown in Fig. 117. It will be evident that there will still be no lateral slipping of the rear wheels, since the force available is insufficient to overcome the friction.

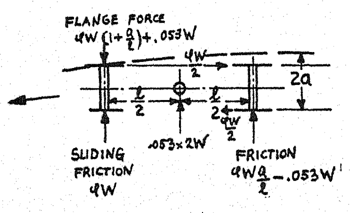


FIG. 117

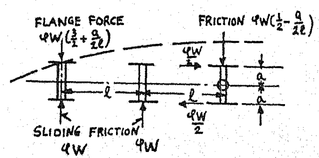


FIG. 118

Now consider a symmetrical 3-axle truck as in Fig. 118. Taking moments about the center of the rear axle 0, the flange force at the leading wheel must balance the friction due to sliding the first and second axles laterally and rotating the rear axle about its center.

The flange force is, therefore

$$\frac{3}{2}\varphi W + \varphi W \frac{a}{2l} = \varphi W \left(\frac{3}{2} + \frac{a}{2l}\right)$$

and the lateral reaction at the rear axle is

$$\varphi W \left(\frac{3}{2} + \frac{a}{2l}\right) - 2\varphi W = -\varphi W \left(\frac{1}{2} - \frac{a}{2l}\right)$$

If the centrifugal force and elevation are taken into account at the

maximum speed, there will be an additional outward force of $0.053 \times 3W$ at the center, resisted at the two outer wheels, since the center axles are slipping and cannot resist any more force.

The flange force at the front wheel therefore becomes

$$\varphi W\left(\frac{3}{2}+\frac{a}{2l}\right)+\frac{3}{2}W\times\cdot 053$$

and the reaction at the rear axle is

$$\varphi W\left(\frac{1}{2}-\frac{a}{2l}\right)+\frac{3}{2}W\times\cdot 053$$

Normally $\frac{a}{l}$ is of the order of $\frac{1}{2}$ and this reaction is not sufficient to slip the rear wheels, since it amounts to but 12% of the weight, which is less than the coefficient of friction.

For speeds higher than normal, however, the rear wheels may slip. Assuming that $\frac{a}{l}=\frac{1}{2}$, then the rear wheels will slip if the reaction due to the centrifugal force exceeds the coefficient of friction under these conditions.

Example. If the curvature is 2°, or $R = 2865$ ft. and the elevation of the outer rail is 5 in., let the speed be V m.p.h.

The net outward force divided by the weight is

$$\frac{\left(\frac{88V}{60}\right)^2}{2865 \times 32.2}-\frac{5}{56\frac{1}{2}}$$

This reaches 5.3% at 77 m.p.h. At 83 m.p.h. it increases to 7.7%, and the reaction at the rear axle is

$$(\tfrac{3}{2} \times 7.7) + 3.7 = 15.2\%$$

which is slightly above the 15% usually assumed for the coefficient of friction for such speeds.

Therefore, at 83 m.p.h. the rear axle would slip outwards until its outer wheel flange touched the outer rail. When this happens, it will be seen that the middle axle must be running radially and no longer

has to slip sideways. But if this friction force is no longer present on the middle axle, the reaction at the rear wheel will decrease by an amount $\frac{1}{2}\varphi W$ and will no longer be sufficient to overcome friction and the rear wheel flange will leave the rail. When it does so, the middle axle will have to slip sideways again and the rear wheel will be pushed back against the outer rail.

The transitions will be more gradual than indicated above, but the result is that the truck will become unstable in the neighborhood of 83 m.p.h. under the conditions given and the rear end will swing back and forth. At sufficiently higher speeds the centrifugal force will reach a value sufficient to hold the truck steadily against the outer rail and it will again become stable. A similar calculation for the 2-axle truck, assuming $\frac{a}{l} = \frac{1}{3}$ which is a comparable value, shows that the unstable speed is about 111 m.p.h. as compared with 83 m.p.h. for the 3-axle truck. Proceeding now to a 4-axle truck as in Fig. 119: taking moments about 0 the flange force on the leading wheel is $\varphi W\left(2 + \frac{a}{3l}\right)$ and equating the sum of the lateral forces to zero, the reaction at the rear axle is $\varphi W\left(1 - \frac{a}{3l}\right)$ which is only slightly below the limiting friction φW.

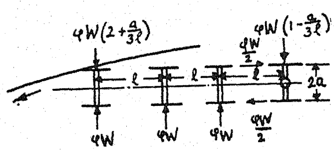


FIG. 119

This wheelbase therefore tends to be unstable with the rear end swinging from side to side. It will not become really stable until the centrifugal force is sufficiently great to slip both the rear and third axles laterally and hold the two outer wheel flanges definitely against the outer rail. Thus there will be a wide range of instability.

In the examples given above it has been assumed that the curve is of sufficiently large radius so that there are no contacts between wheel flanges and the inner rail. In many cases there is flange contact with both inner and outer rails. In Fig. 120AB is the center line of a truck in the position in which it would run (with wheel B radial) if there were no flange contact with the inner rail. Due to the inner rail contact, however, wheel B is actually at B'. A little consideration will show that such a truck runs stably or unstably

in the same ranges of speed as it would if running freely with flange contact at A only. There may be considerable differences, however, when combinations of trucks are concerned. For example, suppose we have a Pacific locomotive as shown in Fig. 121. Here the leading driver, A, is shown against the outer rail and the rear driver, B, against the inner rail. Both leading and trailing trucks are displaced from their central positions. There are now many possible variations. On

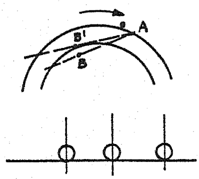


FIG. 120

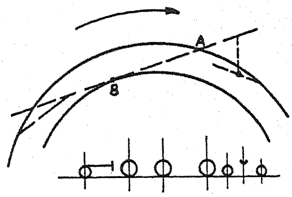


FIG. 121

a longer radius curve the trailing truck may return to the center position if its restraining force is high enough. With sufficiently high restraint in the leading truck the leading driver may be pulled away from the outer rail. Again on sharper curves the flange contact with the inner rail may pass from the rear driving axle to the center driver. As an example of instability let us consider the leading driver, A. In many locomotives of this kind the leading truck is provided with a restraining device, such as a swing bolster, which opposes lateral motion of the truck with respect to the main frame with a force increasing with the displacement. Suppose that, for the condition shown in Fig. 121, the restraint is sufficient to pull the leading driver A away from the outer rail. Suppose further that when the driver A reaches some intermediate position between the rails the forces balance. This may happen because, as the driver moves towards the inner rail, the leading truck displacement gets smaller and hence the lateral force acting on the main frame at the truck center plate decreases. In this equilibrium position, therefore, the front driver must run somewhere between the inner and outer rails, slipping continuously. But a small increase in the coefficient of friction will carry the driver to the outer rail, while a small decrease will carry it to the inner rail. Friction is actually quite variable and the wheel will

undoubtedly run to the outer rail, slip back to the inner rail, run back to the outer rail, slip back to the inner rail and continue an oscillation which under appropriate conditions may be violent enough to deform the track.

In general a necessary condition for stable running on a curve is that the locomotive frame should be definitely located by two flange contacts with reasonable pressure on each and sufficiently far apart to provide a good base for holding the engine. When one of the points is a sliding wheel or a restraint device it will almost always prove unstable. The flexibility of rail and locomotive frame will often result in one "point of flange contact" being distributed over two or more adjacent wheels, but this does not affect the general principle that two such points or regions of flange contact must exist rather far apart compared to the length of the engine.

The study of motion on curves is very easy if a diagram is used in which the lateral motions can be clearly shown to a large scale, preferably full size. One of the best diagrams for this purpose is described below.

A commonly used scale for laying out locomotives and cars is $\frac{1}{16}$ or $\frac{3}{4}$ inch to the foot. If a layout on a curve is made to this scale, the lateral clearances will be so small that they are difficult to see and accurate measurements of lateral motion are impossible. Now suppose the layout is made on a flexible sheet and that the sheet is stretched to 16 times its original size in the direction at right angles to the direction of the track at some point near the middle of the locomotive. Then the length of the locomotive on the layout is not changed, but the lateral clearances are brought back to approximately full size.

In actually drawing the diagram it is plotted using $\frac{1}{16}$ scale for horizontal lines and full scale for vertical lines on the paper which is exactly equivalent to stretching the $\frac{1}{16}$ scale layout. It will be seen readily that the diagram will be perfectly accurate, since no approximations are used in its construction. Experience in using the diagram will soon show that many approximate short cuts are possible without appreciable inaccuracy. However, any approximation which is made can always be checked by using the diagram in a strictly accurate way. The key to using the diagram successfully is to remember

that all measurements must be made vertically or horizontally to the appropriate scales.

The center line of uniformly curved track is a circle and if a circle is stretched in one direction it becomes an ellipse. The diagram is, therefore, called an ellipse diagram, because the center line of the track is usually part of an ellipse.

Construction of the Ellipse Diagram. In the following discussion the diagram will be $\frac{1}{16}$ scale horizontally and full scale vertically.

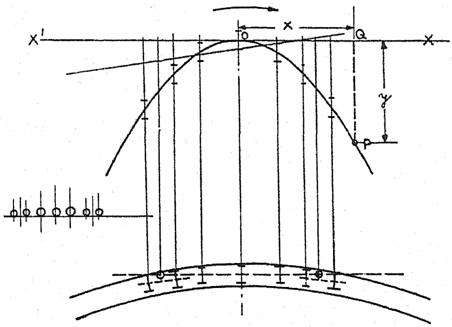


Fig. 122

Any other convenient scales may be used in exactly the same way and the choice is merely one of convenience.

First make a $\frac{1}{16}$ scale layout of the locomotive on the curve as shown in the lower part of Fig. 122. Then project vertical lines up through the centers of all the axles and through any other points of interest, such as the truck center plates, etc.

Draw a horizontal line $X'OX$ near the top of the diagram to represent a tangent to the center line of the track. The curve may now be laid in. Take for example any point P on the curve. On the original circle (see Fig. 123) $QP = OS = \dfrac{SP^2}{ST} = \dfrac{OQ^2}{2R - QP}$ where R is the radius of the circle.

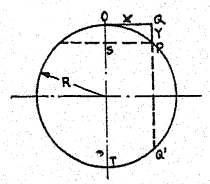


Fig. 123

$$\left(\text{Or more directly } QP \cdot QQ' = QO^2 \quad \therefore \quad QP = \frac{OQ^2}{2R - QP}\right)$$

Therefore, $$2R \times QP - QP^2 = OQ^2$$

$$\therefore \quad QP = \frac{OQ^2}{2R} + \frac{QP^2}{2R}$$

or, putting $$OQ = x, \qquad QP = y$$

$$y = \frac{x^2}{2R} + \frac{y^2}{2R}$$

Since y is small compared with x, this is the most suitable form for calculation, for example if

$$x = 30 \text{ ft.} \qquad R = 900 \text{ ft.}$$

$$y = \frac{30^2}{1800} + \frac{y^2}{1800}$$

1st approximation $$y = \frac{30^2}{1800} = \frac{900}{1800} = \frac{1}{2} = .500 \text{ ft.}$$

2nd approximation $$y = \frac{1}{2} + \frac{(\frac{1}{2})^2}{1800} = .50014 \text{ ft.}$$

3rd approximation $$y = \frac{1}{2} + \frac{(.50014)^2}{1800} = .50014 \text{ ft.}$$

It is clear that in this case the first approximation is about as close as can be drawn and the second approximation is perfectly accurate for all practical purposes. By setting a slide rule or calculating machine at $\frac{1}{2R}$, all terms, including all corrections can be read off.

To plot P on the ellipse diagram we have OQ is 30 ft., which to $\frac{1}{16}$ scale is $22\frac{1}{2}$ in., or preferably use a $\frac{1}{16}$ scale direct. QP is .50014 ft. = 6.002 in.

Usually it is unnecessary to calculate any ordinates of the ellipse except at the axles, center plates, etc. Intermediate sections can be drawn freehand.

At each axle the lateral clearances should now be indicated on the vertical lines through the axle center lines. On each side of the track

center line mark off half the total lateral play of the flanges in the track and also half the total lateral play of the axle in the frame. For example, in Fig. 124 PQ is a part of the ellipse diagram and represents the center line of the track, AB is the vertical line through the center of an axle, then TT' is the total clearance of the flanges in the track and $ST = S'T'$ = half the "internal clearance"; that is, the clearance

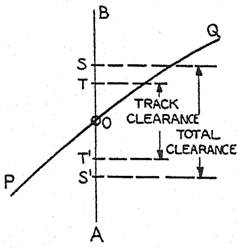


FIG. 124

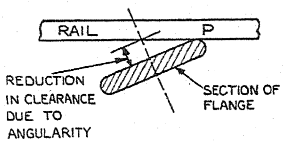


FIG. 125

between the axle and the frame. In calculating TT', the track clearance, it is sometimes thought necessary to allow for the displacement of flange contact due to the angle between the wheel and the rail. For instance, in Fig. 125 a plan view of the rail is shown, with a section through the wheel flange. The point of contact is actually at P and, because of the relative angularity, there is a reduction in the effective track clearance of the axle in the track. This effect is not important except on relatively sharp curves.

After the ellipse and clearances are laid out, the locomotive center line is laid in according to some possible condition in which no clearances are exceeded. The forces are determined and if all parts of the locomotive are found to be in equilibrium the position has been chosen correctly. If some flange pressure is negative or there is any other sign that the condition is not one of equilibrium, the position of the locomotive should be changed to correct it. A little experience will enable equilibrium positions to be located at the first or second attempt in almost all cases. It may be found that no equilibrium position exists and then the locomotive will be expected to oscillate. The process is best studied by working through a few typical cases.

Chapter 20

FREQUENCY OF VIBRATION OF LOCOMOTIVES

The behavior of steam locomotives usually is controlled by forced vibration due to the reciprocating parts. This has a frequency equal to that of rotation of the driving wheels, so that

$$f = \frac{V \times 5280}{3600} \times \frac{12}{\pi \times D}$$

$$= 5.62 \frac{V}{D} \text{ cycles per sec.}$$

where V = speed in m.p.h.

D = wheel diameter in in.

It is usual to design steam locomotives so that the wheel diameter in inches is fairly near to the maximum speed in m.p.h., so that $\frac{V}{D}$ is not very different from 1.0 at the maximum speed. Hence the forced vibration at the maximum speed will be roughly of the order of 5 cycles per sec., which is fast enough to produce considerable damping.

The natural periods of vibration are more difficult to measure or to calculate because of the variety of motions which can take place under different conditions. The natural periods are particularly important on electric locomotives and similar vehicles where there is no forced vibration to break up the natural ones.

There are three main factors which influence frequency of vibration. First is the natural frequency of roll of the body on the springs, second the effect of coned wheels and rail joints, and third the swinging of the wheelbase in the track. This last may be visualized by imagining a rigid wheelbase with its front wheel running steadily against the left rail. If the rear wheel is against the right rail, the wheelbase

will swing across the track like a damped pendulum, pulled by the friction and creepage forces at the wheel treads. When the whole wheelbase reaches the left rail the front wheel will move across the clearance to the right rail and the wheelbase will again swing after it. This is of course a simplified picture of what happens, but is convenient in separating the different factors.

The first and third factors—that is, roll and swinging in the clearance—are pendulum-like motions which tend to have a constant frequency independent of speed. The second factor, rail joints and coned wheels, tends to produce oscillation in a constant distance and therefore a frequency which increases with speed.

Observation shows that on large locomotives the frequency is generally nearly independent of speed except that at some speeds the motion changes to one having a definitely different frequency. The lowest frequency is usually the most important and this may be said to be generally independent of speed. Cars with short light trucks are usually more influenced by wheel contours and rail joints and have frequencies which vary more with speed.

The following records of observed frequencies will give some indications of what is to be expected.

Locomotive 2–B–2. (Fig. 126.) The lowest frequency is about 1.10 cycles per sec., which was observed at 40 m.p.h. and also at 95 m.p.h. This is probably controlled by the rolling of the body on the main driver springs, the roll being very pronounced at the higher speeds. At intermediate speeds a higher frequency motion can be maintained which is most noticeable at about 1.80 cycles per sec. at 75 m.p.h. This is probably due to rapid oscillation in the track clearance which can sustain itself but is not violent enough to cause the violent roll which lengthens the periodic time of an oscillation.

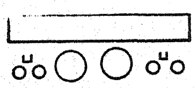


FIG. 126

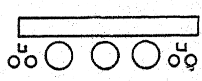


FIG. 127

Locomotive 2–C–2. (Fig. 127.) The lowest frequency is about 1.20 cycles per sec., observed particularly at 90–105 m.p.h. On rough track the same period can be observed at lower speeds, such as 70

m.p.h. High resistance trucks increase the frequency slightly; for example, under certain conditions where the frequency is 1.25 for low resistance trucks, it is 1.50 for high resistance trucks.

The frequency is sometimes reduced by increasing the flexibility of the springs. For example with a static deflection of the main driver springs of about 3 in. under some conditions the frequency is approximately 1.80. If the static deflection is increased 2 or 3 times the frequency returns to 1.20. On rough track, where the roll increases, the frequency drops further to about 1 cycle per sec. It is fairly clear that if the springs are made either very stiff or very flexible, the natural frequency of roll becomes very different from the other natural frequencies and has less effect than if the frequencies are closer together.

Locomotive 2–D–2. (Fig. 128.) The locomotive shows a greater steadiness than the shorter ones, as would be expected. With very stiff springs the frequency is 1.10–1.15 cycles per sec. at 90–120 m.p.h.

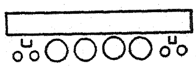


FIG. 128

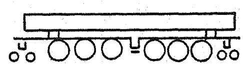


FIG. 129

Locomotive 2 – C + C – 2. (Fig. 129). For high speed this type of engine is fitted with stabilizing devices which prevent relative motion of the various trucks except when track forces are sufficient to overcome the setting of the stabilizers. Therefore on straight track the locomotive oscillates as if it had one single rigid wheelbase and is very stable.

A typical frequency is about 1.25 cycles per sec. at 90 m.p.h. This frequency is quite constant with speed over a considerable range, but at higher speeds the frequency increases slightly, so that it may be 1.50 at 115 m.p.h.

Experience shows that the stabilizer which restrains the main driving trucks, usually by connecting the articulated joint to the body, has very little if any effect on the frequency. The guiding truck stabilizers, however, have a great effect and if they are omitted the frequency rises to about 2 cycles per sec. A very critical oscillation has been observed at exactly 71 m.p.h. on a locomotive of this type, with a frequency of

about 3.2 per sec., which is evidently due to rail joints, since this is the frequency of passing joints at this speed. Such an oscillation, however, is rare and is very small, so that instruments are required to detect it.

The most satisfactory way to find the frequency of a locomotive is to measure it on the locomotive itself or on one of similar construction. This is often practical. For example, if there is a question of the maximum safe speed or the effect of some change in clearance or restraint, this may not affect the frequency and actual observations may be used. It is possible also to make detail calculations of the motion of a locomotive, but they are long and laborious and will still contain some uncertainties, so that they need seldom be made except as a matter of general research into locomotive motion.

If it is necessary to estimate the frequency of a locomotive which cannot be observed, the most practical way is to base the estimate on observations of similar locomotives. The following rough rules may sometimes be found helpful:

(1) The frequency of roll is less than would be calculated for purely rolling motion because the locomotive has some lateral freedom which is not considered in calculations of pure roll. The following rule of thumb is useful if used with caution:

Let a = static deflection of driver spring system in in.
k = fraction of the total spring supported load which is carried by the driver spring system.
f_r = approximate frequency of rolling motion,

Then $f_r = 2.35\sqrt{\dfrac{k}{a}}$

(2) The nosing frequency, if roll is prevented, is approximately

$$f_n = \frac{1}{2\pi}\sqrt{\frac{\Sigma fx \times g}{I}}$$

where f = creepage coefficient of each axle
$= 3500\sqrt{\text{axle load lb.} \times \text{wheel diam. in.}}$ in lb.
x = distance of the axle behind the leading axle, in ft.
Σ represents summation over all axles

g = gravity = 32.2 ft/sec.^{2}
I = moment of inertia of the locomotive, around a vertical axis through the leading axle, in lb. ft.^{2}

(3) The resultant frequency is given approximately by the average of f_r and f_n, except that for locomotives which roll considerably, f_r is a better approximation and for articulated locomotives, without truck stabilizers, $1.30 f_n$ is a better approximation.

Example. Let us consider an articulated locomotive similar to that in Fig. 129.

a = static deflection of driver springs = 3.9 in.
k = fraction of total spring-supported load which is carried by the driver spring system = 0.63

$$\text{Therefore } f_r = 2.35\sqrt{\frac{k}{a}} = 2.35\sqrt{\frac{0.63}{3.9}} = 0.94 \text{ cycles per sec.}$$

For the truck wheels the creepage coefficient f is 4,200,000 lb. For the driving wheels the creepage coefficient is 5,900,000 lb. The sum of fx taken over all the wheels is

1,800,000,000 lb. ft.

The moment of inertia of the locomotive about a vertical axis through the leading axle is

648,000,000 lb. ft.^{2}

Therefore the nosing frequency f_n is

$$f_n = \frac{1}{2\pi}\sqrt{\frac{\Sigma fx \times g}{I}}$$

$$= \frac{1}{6.28}\sqrt{\frac{1800 \times 32.2}{648}}$$

$$= 1.51 \text{ cycles per sec.}$$

Therefore the resultant frequency may be expected to be approximately

$$\frac{f_r + f_n}{2} = \frac{0.94 + 1.51}{2} = 1.23 \text{ cycles per sec.}$$

If the track is rough and the engine begins to roll considerably, the frequency will be approximately

$$f_r = 0.94 \text{ cycles per sec.}$$

If the stabilizers are removed from the trucks, the engine will nose much more than it will roll and the frequency will be approximately

$$1.3 f_n = 1.3 \times 1.51 = 1.96 \text{ cycles per sec.}$$

These values actually check closely with observations on locomotives of this general type.

In addition to frequency, it may be necessary to estimate the damping of a locomotive which is oscillating. Observations indicate that a logarithmic decrement of about 1.0 will apply to a flexible locomotive and about 2.0 to a fairly rigid locomotive. As particular examples, a long articulated locomotive with low resistance trucks and no stabilizers had a decrement of 0.9. The same locomotive with high resistance trucks and stabilizers had a decrement of 2.2.

These indicate considerable damping, much of which is probably in the spring suspension. A decrement of 1 means that in a free vibration each swing is 37% of the preceding one. A decrement of 2 means that each swing is 13% of the preceding one, the relation being

$$\delta = \log_e \frac{y_1}{y_2} = 2.3 \log_{10} \left(\frac{y_1}{y_2}\right)$$

where δ = logarithmic decrement

y_1 and y_2 are successive swings.

These values were obtained when the locomotives were rolling violently on rough track. On smooth track the roll decreases and the decrement may be much less. The above figures and examples all refer to locomotives with spring suspensions and journal boxes outside the wheels. Steam locomotives with inside journals usually have considerably lower frequencies of roll.

Chapter 21

RAILWAY PASSENGER CARS

There has been much difficulty in establishing general principles for the design of railway passenger cars which are free from vibration. Designs which are satisfactory in one place are often unsuccessful under different conditions. The most common cause of vibration is probably poor maintenance, which results in interferences of moving parts, use of incorrect springs, failure to line up trucks accurately, wheels out of round, etc. These conditions require no comment here. It has been established that the riding of a car is greatly affected by the behavior of the cars next to it. A test car, run next to a series of other cars on a succession of runs over the same track showed riding qualities varying very widely according to the car next to it. When the next car was a good rider the test car was also good. When the next car was a poor rider the test car rode very poorly and in fact the behavior of the test car as recorded by instruments was a fair measure of the riding qualities of the car next to it. This is a fact of great importance in the testing and observation of cars for riding qualities and also shows the need for having every car in a train a good rider. A single bad car may upset the surrounding ones.

With cars in good order the chief features affecting oscillation are the springs and hangers, the truck wheelbase and centerplate spacing and the contour of the wheel treads.

Spring Systems. Almost every conceivable combination of springs has been used in passenger car trucks and no attempt will be made to describe them here, but one example will be chosen to illustrate the more important features.

Fig. 130 shows a truck built for passenger cars by the Pullman-Standard Car Manufacturing Company. These trucks or varieties of similar design have been used very successfully in lightweight, high

speed trains. The plan and the elevation of the truck are shown in Fig. 131. The truck frame is supported from the journal box housings through large helical springs, two for each journal box, which are clearly shown in the illustrations. The bolster is supported through full elliptic leaf springs on long inclined swing hangers as shown in Fig. 132. These hangers are supported in turn by an intermediate bolster, shown in Fig. 133. The intermediate bolster is hung from

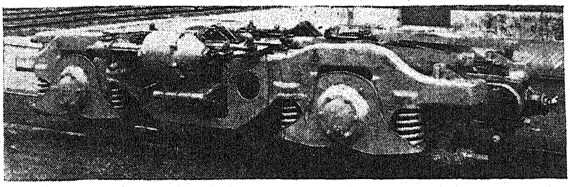


FIG. 130

the main truck frame through short vertical hangers and helical springs.

It will be seen that this arrangement puts light undamped springs close to the journals, where they are most effective, and puts flexible, damped springs directly under the bolster. In addition, the intermediate spring-supported bolster adds flexibility and cushions any sudden forces in the main swing hangers.

The majority of trucks do not have intermediate bolsters and the swing hangers are fastened directly to the main frame, but the intermediate bolster was included in this truck in order to obtain better riding qualities at high speed with light equipment.

Spring Flexibility. The deflection of car springs varies considerably. In general the greater the flexibility the smoother the riding. Good riding cars have been built with a static spring deflection of less than 6 in. Tests indicate that improvement can be obtained by increasing this to about 12 in. but greater deflections than this have not shown appreciably better characteristics. A spring with a static deflection of "d" inches has a natural frequency.

$$F = \frac{1}{2\pi}\sqrt{\frac{32.2 \times 12}{d}} = \frac{3.127}{\sqrt{d}} \text{ cycles per sec.}$$

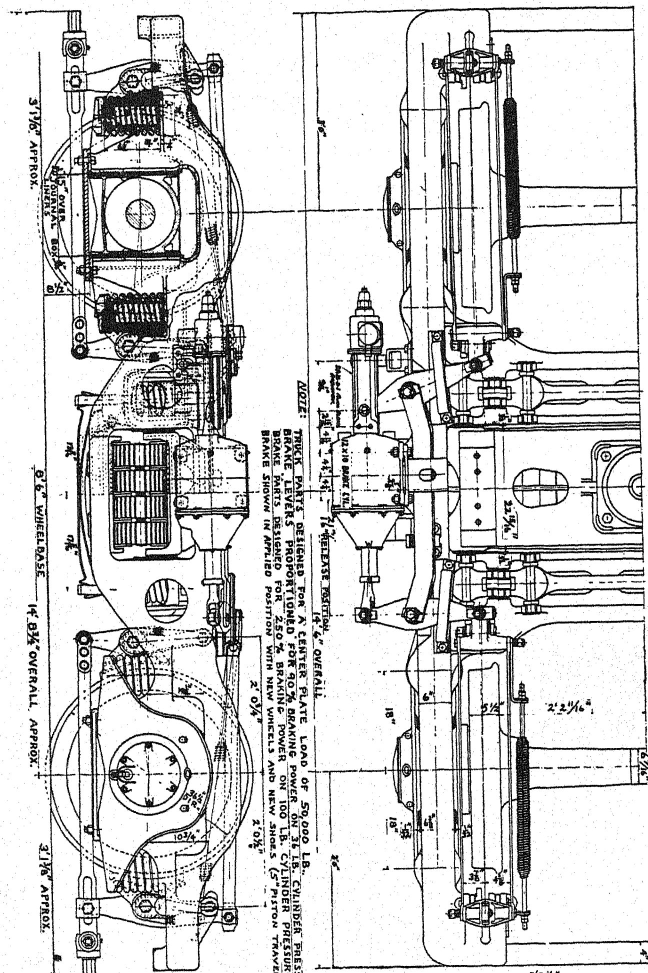
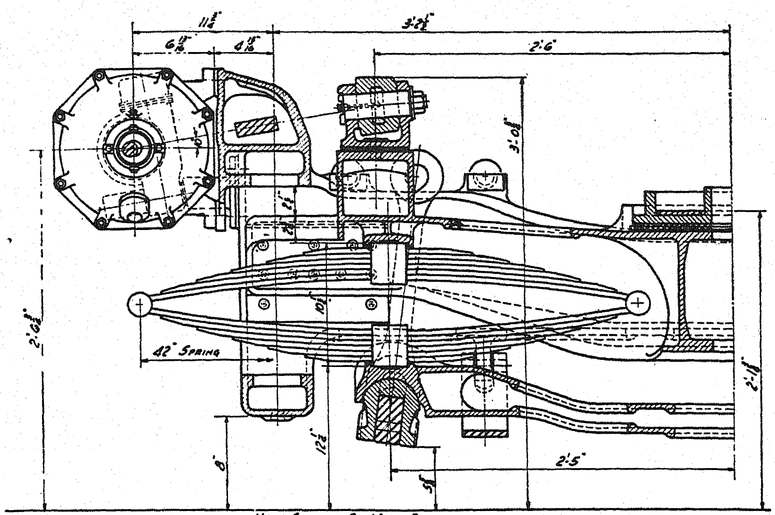
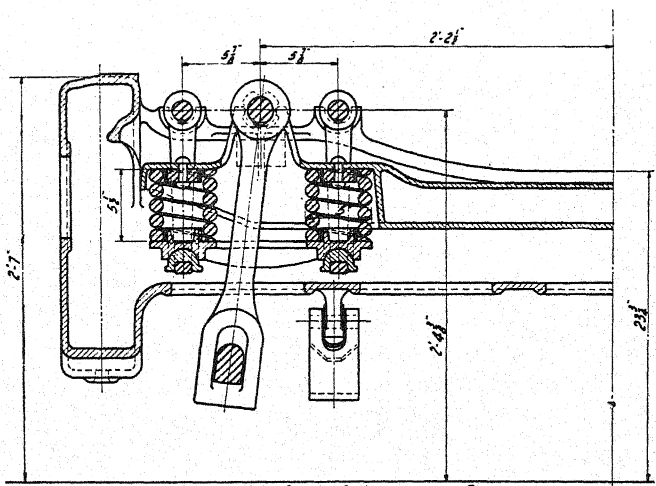

NOTE:
TRUCK PARTS DESIGNED FOR "A" CENTER PLATE LOAD OF 50,000 LB.
BRAKE LEVERS PROPORTIONED FOR 90% BRAKING POWER ON 36 LB. CYLINDER PRESSURE
BRAKE PARTS DESIGNED FOR 250% BRAKING POWER ON 100 LB. CYLINDER PRESSURE
BRAKE SHOWN IN APPLIED POSITION WITH NEW WHEELS AND NEW SHOES (5" PISTON TRAVEL)
14'6" OVERALL
8'6" WHEELBASE
14'8¾" OVERALL APPROX.

FIG. 131



HALF SECTION AT MAIN BOLSTER

FIG. 132



HALF SECTION AT INTERMEDIATE BOLSTER

FIG. 133

Hence springs with 3, 6, and 12 in. static deflection have natural frequencies:

Static Deflection Inches	*Natural Frequency Cycles per Second*	*Natural Period Seconds*
3	1.81	0.55
6	1.28	0.78
12	0.91	1.10

At high speeds the irregularities of track have considerably higher frequencies; for example at 90 m.p.h. the frequency of joints in a 39 ft. rail is

$$\frac{132}{39} = 3.4 \text{ per sec.}$$

Most track irregularities are shorter than a rail length and hence the natural frequency of car springs will be considerably less than that of the disturbances, explaining why greater flexibility is desirable. This also indicates that the spring damping should be as small as is consistent with damping of occasional large oscillations.

The oscillations have the same general character as those of a spring-supported weight rolling over a wavy track. If these waves are uniform with a height "h" in. above and below the average it is shown above that the weight moves in a similar wavy path, but the amplitude is

$$h \times \frac{1}{1 - \left(\frac{f}{f_0}\right)^2}$$

where f = frequency with which the weight passes over the waves and f_c is the natural frequency of vibration of the weight on its supporting spring.

The acceleration of the weight has a maximum value of

$$\pm h \frac{1}{1 - \left(\frac{f}{f_c}\right)^2} \times 4\pi^2 f^2$$

When the frequency of the waves is considerably greater than the natural frequency of the weight,

$$1 - \left(\frac{f}{f_c}\right)^2 \text{ is nearly equal to } - \left(\frac{f}{f_c}\right)^2$$

so that the maximum acceleration is approximately

$$4\pi^2 h f_c^2$$

But the natural frequency is

$$f_c = \frac{1}{2\pi}\sqrt{\frac{g}{a}}$$

where "a" = static deflection of the spring under the load, hence the maximum acceleration is approximately

$$g\frac{h}{a}$$

This indicates that for high speed passenger cars, when the speed is high enough so that track irregularities are short compared to the natural periods of vibration on the springs, the accelerations of the car body vary with the roughness of the track, and inversely as the spring deflection and are not affected by further increase of speed. This is found to be the case and vibrations which increase with speed at high speeds are primarily due to nosing, internal motions, etc., and not to forced vibrations from rough track.

Rolling. If a car is built with the extremely flexible springs which are desirable for easy riding at high speed it may be found that it rolls excessively, particularly on curves. In fact, cars have been tested in which the springs were so flexible that the car body would roll towards one side on a curve and fail to right itself on reaching straight track.

Tests have been made to find the effect of preventing this excessive roll and it has been found that the roll can be reduced without losing the good effects of the flexible springs.

The measures which reduce roll without affecting the vertical flexibility of the springs are:

(1) Lowering the center of gravity.
(2) Increasing the distance between springs from one side of the truck to the other.
(3) Adding anti-roll stabilizers.

A low center of gravity reduces the overturning moment on the car due to centrifugal force on curves and so reduces the tendency of the car to roll and to overturn. It is therefore a very important requisite of high speed cars.

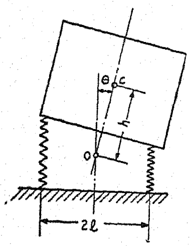


FIG. 134

Wide spring centers, obtained with many outside frame trucks, may give sufficient stability to make anti-roll stabilizers unnecessary.

Consider a simplified car, shown in Fig. 134 in which O is the center of rotation of the car body, fixed by the truck construction, C is the center of gravity of the rolling parts.

$2l$ = distance apart of the springs.
k = stiffness of the spring system on one side in lb. per in. deflection.
θ = angle of roll of the car body.
h = height of center of gravity above center of oscillation.
K = radius of gyration of the rolling parts about the center of gravity, C.
w = weight of the rolling parts.

Then, the moment of resisting rotation is

$$(2kl^2 - wh)\theta \quad . \; . \; . \; . \; . \; . \; . \; . \quad (41)$$

It will be seen that the resisting moment increases faster than the square of the spring base, so that doubling the spring base will provide more than four times the righting moment. This shows the disadvantage of using inside frames or closely spaced bolster springs. If these are necessary for other reasons on a high speed car, it probably will be desirable to use an anti-roll device with flexible springs.

Two types of anti-roll stabilizers will be described to illustrate the principle. In Fig. 135 a bolster *A* is carried on the journal boxes *BB* through springs designed to give adequate vertical flexibility. The bolster carries two pivoted equalizers *E* and *F*, which are connected by a spring at *S*, the other ends resting on the journal boxes (or in an actual truck on an equalizing bar which connects two boxes on one side). It will be seen that for pure vertical motion the equalizers *E* and *F* move together and the spring *S* is not affected. For roll, however, the equalizers must move relatively at *S* and roll is resisted by the spring *S* as well as by the main journal springs. In this way the resistance to roll may be increased to any desired amount by suitably designing the spring *S*.

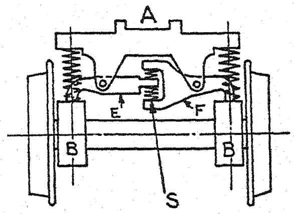


FIG. 135

A second type is particularly suitable for spring bolster trucks.

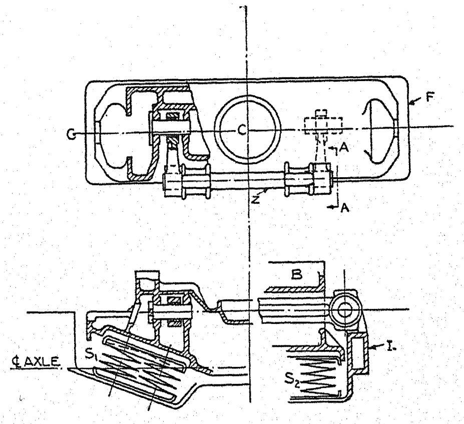


FIG. 136

In Fig. 136 the truck frame *F* rests on the spring system S_1. Hangers support an intermediate member *I* which supports the bolster *B* through a second spring system S_2. Torsion bars *Z* join the bolster *B* to the intermediate member *I*, forcing them to remain parallel and

hence limiting the action of the springs S_2 to purely vertical motion. The full flexibility of both S_1 and S_2, in series, are available for vertical motion which roll is resisted by the system S_1 only, and the flexibility in roll is therefore reduced by the stabilizers Z. The figures are of course purely diagrammatic, but the application to actual trucks will be understood easily.

Synchronism. Railroad car vibrations are often very much affected by some relationship between bolster spacing, truck wheel-base, wheel diameter and rail joint spacing. Two or more of these factors may be so related that a regular disturbing force is easily set

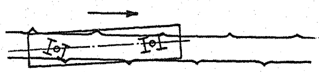


FIG. 137

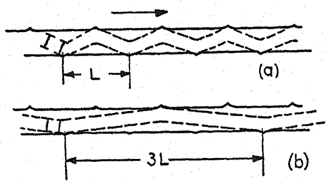


FIG. 138

up which can cause considerable vibration. As a simple illustration, if track has joints spaced alternately in the two rails and if a car has its truck bolsters spaced $1\frac{1}{2}$ rail lengths apart, it is easy to build up a regular nosing oscillation. Fig. 137 shows a plan view of such a condition. The rail joints are weaker than the solid rail and are shown as displaced outward from the line of the rail. With 39-ft. rails the condition would occur with a bolster spacing of approximately $39 \times 1\frac{1}{2} = 58\frac{1}{2}$ ft. Now the trucks may hit every rail joint as in Fig. 138 (a) or they may hit every third joint as in Fig. 138 (b).

In Fig. 138 (a) the frequency of oscillation—that is, the number of times per second the truck goes across the track and back again—is

$$f = \text{frequency} = \frac{\text{speed in ft. per sec.}}{\text{rail length in ft.}}$$

$$= \frac{\text{speed in m.p.h.}}{\text{rail length in ft.}} \times \frac{88}{60}.$$

In the case of Fig. 138 (b) the frequency is clearly only $\frac{1}{3}$ as great,

or
$$f = \frac{\text{speed in m.p.h.}}{3 \times \text{rail length in ft.}} \times \frac{88}{60}.$$

Again taking 39-ft. rails as an example, the frequency is

$$f = \frac{\text{m.p.h.}}{39} \times \frac{88}{60} \quad \text{Fig. 133 } (a)$$

or
$$f = \frac{\text{m.p.h.}}{3 \times 39} \times \frac{88}{60} \quad \text{Fig. 133 } (b)$$

These are shown in Fig. 139. To look for likely vibrations we would estimate the frequency of vibration of the car body swinging on the swing bolsters. For example, a frequency of about 1.5 per sec. would indicate a good possibility of an "every joint" motion in the neighborhood of 40 m.p.h. A frequency of 1 per sec. would reduce this speed to about 27 m.p.h., but would introduce the possibility of an "every 3 rail joint" vibration in the neighborhood of 80 m.p.h.

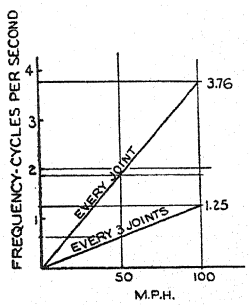


FIG. 139

If the bolster spacing is approximately one rail length, a side sway would be suspected as in Fig. 140.

These are side motions of the trucks, but vertical or fore-and-aft vibrations may also be due to similar causes. These are particularly common with track with opposite rail joints. In this case a bolster spacing very close to a rail length would be favorable to either a vertical vibration of the car or a fore-and-aft vibration. The two types are indicated in Figs. 141

$$\left(\text{Vertical vibration.} \qquad \text{Frequency} = \frac{\text{m.p.h.}}{L} \times \frac{88}{60}\right)$$

and 142

$$\left(\text{Fore-and-aft vibration.} \quad \text{Frequency} = \frac{\text{m.p.h.}}{2l} \times \frac{88}{60}\right).$$

It will be noticed that the fore-and-aft vibration is accentuated particularly if the rail length is an even multiple of the truck wheelbase. For instance, if the truck wheelbase is 8 ft. and the rail length 48 ft., so that $L = 6l$ or if the dimensions are approximately these, a fore-and-aft vibration is to be suspected. In this case if the fore-

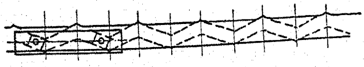


FIG. 140

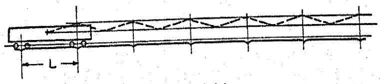


FIG. 141

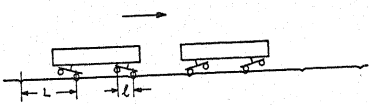


FIG. 142

and-aft frequency of vibration of the car body on the trucks is 4 per sec. the speed at which the vibration would be worst would be in the neighborhood of that given by

$$4 = \frac{\text{m.p.h.}}{2 \times 8} \times \frac{88}{60}$$

or about 44 m.p.h.

It will also be evident that a bolster spacing equal to $\frac{1}{2}$ rail length or $1\frac{1}{2}$ rail lengths, will encourage galloping of the car body on track with oppositely spaced joints.

An interesting case is that shown in Fig. 143.

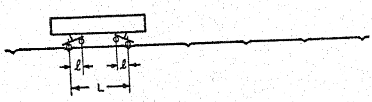


FIG. 143

Here the bolster spacing is equal to a rail length minus the length of one truck wheelbase. The two trucks tend to tilt in opposite directions at the same time. Thus the car body cannot move fore and aft with the truck center plates and violent shocks in the trucks may result, as they push and pull against each other in their action on the car body.

Another condition is a close relationship between the circumference of the wheels and rail length or truck wheel base. For example a wheel $37\frac{3}{8}$ in. in diameter would turn just four times in a 39-ft. rail length. The same applies to a $31\frac{5}{8}$-in. wheel on a 33-ft. rail. If a wheel is eccentric or out of balance under these conditions, the regular rail joint impulses tend to bring the eccentric wheel load into phase with themselves and to hold it in phase. Thus also a truck wheel base equal to the wheel circumference gives further opportunity for successive axles to synchronize their unbalance with rail joint impulses.

With all these possibilities for synchronism it is not surprising that rough riding is often observed on cars and that its cause may be obscure and easily affected by minor changes. In some cases the vibration is due to some fundamental synchronism such as that between bolster spacing and wheelbase, in which case it may be extremely difficult to avoid vibration at least over some range of speeds.

These effects of synchronism are particularly pronounced when heavy traffic of similar cars at about the same speed passes over the track. The track then becomes weakened or deformed at the joints or other places which are repeatedly hit by wheel flanges and this increases the vibration with a cumulative effect. This has been observed for example in some suburban electric service where frequent trains of similar cars operate at uniform speeds. Change in car springs or hangers, or reduction in track gage or change in relative position of rail joints in the two rails, may in some cases improve matters or a slight change in wheel size may be effective if this is an important factor.

In large locomotives with numerous axles it is not generally observed that rail joints or wheel circumferences have any marked effect on vibration but synchronism will of course have some effect on increasing vibration. Vibrations synchronizing with rail joints have been observed on a large locomotive but were small and could only be produced at almost exactly one speed. The slightest observable variation in speed would cause the vibration to disappear.

Swing Hangers. Most passenger car trucks have swing bolsters. That is, the bolster which carries the center plate is hung on links as shown in Fig. 132. The distance apart, length and angularity of these links differ in different designs and have an effect on the opera-

tion. The wider the distance apart, the more stable is the arrangement and the longer are the hangers which can be used. The longer the hangers, the greater is the motion which can be provided for without undue angularity. The effect of the angle of the hangers is more complicated. If the hangers are straight, the body will behave like a pendulum whose length is equal to the length of the hangers. On entering or leaving curves this may allow the body to swing considerably to one side and this flexibility may be too great.

At rough places in the track the truck can move sideways without any appreciable shock on the body and the only effect is to compress the springs slightly.

In order to increase the lateral restraint it is usual to incline the hangers slightly, so that the lower ends are farther apart than the tops. This increases the lateral restraint rather quickly and increases the frequency correspondingly. If two large an inclination is used the restraint will reach a rather critical point, depending on the various dimensions, where it becomes excessive and the links are almost useless. It is also important to note that when the links are inclined the body must rotate as it swings from side to side. It follows that if the truck is suddenly displaced sideways at a rough spot the body must either rotate very suddenly, so that a shock is felt, or the rotation must be cushioned by springs. If these springs are in the hangers themselves, then no shock can be transmitted from the truck to the car body and no shock is produced anywhere in the system except at the rail, where the displacement originates. If there are bolster springs between the hangers and the body there may be minor shocks in the truck due to the inertia of the various parts but these are isolated more or less effectively from the car body by the bolster springs. If the springs are all between the axles and the truck frame then a sudden side motion of the truck will cause a sudden change in angle between the truck frame and the car body and as both of these have inertia a shock will be felt.

It may be concluded that:

(*a*) A swing bolster is used to cushion lateral blows due to rough track and short spirals, at the ends of curves.

(*b*) Long hangers should be used to allow large side motions of the truck to take place without shock to the body.

(*c*) These straight long hangers have too little restraint on curves. Therefore, they are inclined to increase the restraint. The most suitable inclination appears to vary with track and other conditions and varying conclusions have been arrived at on different railroads. The inclination should be as small as is allowable while still providing sufficient restraint.

(*d*) In order to avoid shocks on the body due to sudden side motion of trucks with inclined hangers, springs should be provided, preferably in the hangers or at least in the bolster. Journal box springs, which cushion vertical track irregularities, are not very effective in cushioning these shocks due to rotation of the body on inclined hangers.

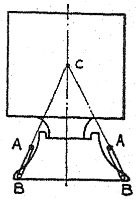


FIG. 144

It is interesting to consider the extreme case where the inclination of the hangers is so great that the intersection of their center lines is at the height of the center of gravity of the body which they support. The condition is as shown diagrammatically in Fig. 144.

The pins AA are fixed in the truck frame. Inclined links hold the the bolster BB and the lines AB intersect at C which is also the height of the center of gravity. Then it will be seen that for very small oscillations C remains fixed, because it is the center of rotation of the body. However much lateral force acts at the center of gravity it will not cause the links to swing because their swinging would not allow the center of gravity to move sideways. If the system oscillates it will do it by the center of gravity oscillating vertically up and down as the body swings on the links. In this case the links are completely ineffective as a cushion against side movement of the truck. Although this condition is extreme, it gives some insight into the factors which affect a practical design which approaches more or less nearly to this condition.

If l = length of each link
$2a$ = spacing of the top pins
H = height of center of gravity above the lower pins
α = angle of inclination of the hangers
g = acceleration due to gravity
f = natural frequency of lateral oscillation

Then the following equation holds approximately for small oscillations and small inclinations of the hangers:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{l}\left(1 + \frac{2(H+l)}{a}\sin\alpha\right)}.$$

This, and several other relations are obtained in the following paragraph.

Mathematics of Inclined Hangers. (Fig. 145.)

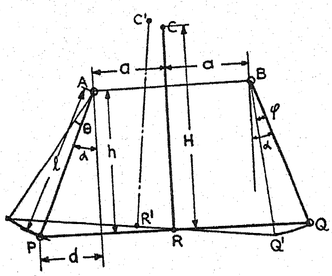


FIG. 145

Let R be the origin of coordinates,

P' is $[-a - l\sin(\alpha+\theta)]$, $[l\cos\alpha - l\cos(\alpha+\theta)]$

Q' is $[+a + l\sin(\alpha-\varphi)]$, $[l\cos\alpha - l\cos(\alpha-\varphi)]$

Now $P'Q' = PQ$

$$\therefore\ [2a + l\sin(\alpha+\theta) + l\sin(\alpha-\varphi)]^2 + [l\cos(\alpha-\varphi) - l\cos(\alpha+\theta)]^2 = [2a + 2l\sin\alpha]^2$$

$$\therefore\ 4al\,[\sin(\alpha+\theta) + \sin(\alpha-\varphi) - 2\sin\alpha] + 2l^2\,[\cos 2\alpha - \cos(2\alpha+\theta-\varphi)] = 0$$

Expanding and disregarding cubes and higher powers of θ and φ,

$$2a\left[(\sin\alpha)\left(1-\frac{\theta^2}{2}\right) + (\cos\alpha)\theta + (\sin\alpha)\left(1-\frac{\varphi^2}{2}\right) - (\cos\alpha)\varphi - 2\sin\alpha\right] + l\left[\cos 2\alpha - (\cos 2\alpha)\left(1-\frac{(\theta-\varphi)^2}{2}\right) + (\sin 2\alpha)(\theta-\varphi)\right] = 0$$

$$\therefore\ 2a\left[(\theta-\varphi)\cos\alpha-\frac{(\theta^2+\varphi^2)}{2}\sin\alpha\right]$$
$$+\,l\,(\theta-\varphi)\left[\sin 2\alpha+\frac{(\theta-\varphi)}{2}\cos 2\alpha\right]=0$$

$$\therefore\ \theta-\varphi=\frac{a\sin\alpha\,(\theta^2+\varphi^2)-\dfrac{l}{2}(\theta-\varphi)^2\cos 2\alpha}{2a\cos\alpha+l\sin 2\alpha}$$
$$=\frac{a\sin\alpha}{2a\cos\alpha+l\sin 2\alpha}(\theta^2+\varphi^2)$$

to the required degree of accuracy.

The coordinates of R' are:

$$\frac{l}{2}[\sin(\alpha-\varphi)-\sin(\alpha+\theta)],$$

$$\frac{l}{2}[2\cos\alpha-\cos(\alpha+\theta)-\cos(\alpha-\varphi)]$$

Expanding, these are:

$$\frac{l}{2}\left[(\sin\alpha)\left(1-\frac{\varphi^2}{2}\right)-(\cos\alpha)\,\varphi-(\sin\alpha)\left(1-\frac{\theta^2}{2}\right)-(\cos\alpha)\,\theta\right],$$

$$\frac{l}{2}\left[2\cos\alpha-(\cos\alpha)\left(1-\frac{\theta^2}{2}\right)\right.$$
$$\left.+(\sin\alpha)\,\theta-(\cos\alpha)\left(1-\frac{\varphi^2}{2}\right)-(\sin\alpha)\,\varphi\right]$$

or
$$\frac{l}{2}\left[-(\cos\alpha)(\theta+\varphi)+(\sin\alpha)\left(\frac{\theta^2-\varphi^2}{2}\right)\right],$$

$$\frac{l}{2}\left[(\sin\alpha)(\theta-\varphi)+(\cos\alpha)\left(\frac{\theta^2+\varphi^2}{2}\right)\right]$$

Let $\lambda=\dfrac{\theta+\varphi}{2}$, then $\dfrac{\theta^2+\varphi^2}{2}=\lambda^2$

to the required degree of accuracy.

Then
$$\theta - \varphi = \frac{2a \sin \alpha}{2a \cos \alpha + l \sin 2\alpha} \lambda^2$$

R' is
$$\frac{l}{2}[-2\lambda \cos \alpha], \quad \frac{l}{2}\left[\frac{2a \sin^2 \alpha}{2a \cos \alpha + l \sin 2\alpha} \lambda^2 + \cos \alpha \lambda^2\right]$$

or
$$-l \cos \alpha \cdot \lambda, \quad l\lambda^2 \left(\frac{a + l \sin \alpha \cos^2 \alpha}{2a \cos \alpha + l \sin 2\alpha}\right)$$

The slope of $P'Q'$ is

$$\frac{l \cos (\alpha - \varphi) - l \cos (\alpha + \theta)}{2a + l \sin (\alpha - \varphi) + l \sin (\alpha + \theta)}$$

$$= \frac{(\cos \alpha)\left(1 - \frac{\varphi^2}{2}\right) + (\sin \alpha)\, \varphi - (\cos \alpha)\left(1 - \frac{\theta^2}{2}\right) + (\sin \alpha)\, \theta}{\left(\frac{2a}{l}\right) + (\sin \alpha)\left(1 - \frac{\varphi^2}{2}\right) - (\cos \alpha)\varphi + (\sin \alpha)\left(1 - \frac{\theta^2}{2}\right) + (\cos \alpha)\, \theta}$$

$$= \frac{(\sin \alpha) \times 2\lambda}{\left(\frac{2a}{l}\right) + (\sin \alpha)(2 - \lambda^2) + \left(\frac{2a \sin \alpha \cos \alpha}{2a \cos \alpha + l \sin \alpha}\right)\lambda^2} = \frac{l \sin \alpha}{a + l \sin \alpha} \cdot \lambda$$

to the required approximation.

The coordinates of C' are:

$$\left[-l \cos \alpha\lambda + \frac{Hl \sin \alpha}{a + l \sin \alpha} \cdot \lambda\right],$$

$$\left[l\lambda^2 \frac{a + l \sin \alpha \cos^2 \alpha}{2a \cos \alpha + l \sin \alpha} + H\left\{1 - \frac{l^2 \sin^2 \alpha}{(a + l \sin \alpha)^2} \frac{\lambda^2}{2}\right\}\right]$$

or
$$-l\lambda \left\{\cos \alpha - \frac{\frac{H}{a} \sin \alpha}{1 + \frac{l}{a} \sin \alpha}\right\},$$

$$H + \frac{l\lambda^2}{2}\left\{\frac{2 + \frac{2l}{a} \sin \alpha \cos^2 \alpha}{2 \cos \alpha + \frac{l}{a} \sin 2\alpha} - \frac{\frac{H}{a}\frac{l}{a} \sin^2 \alpha}{\left(1 + \frac{l}{a} \sin \alpha\right)^2}\right\}$$

Let m = mass of the oscillating portion
k = radius of gyration about the center of gravity
f = natural frequency of oscillation
$\omega = 2\pi f$

Then, equating the potential energy at the end of a swing to the kinetic energy at the mid-point of a swing:

$$mg\frac{l\lambda^2}{2}\left\{\frac{2\times\frac{2l}{a}\sin\alpha\cos^2\alpha}{2\cos\alpha+\frac{l}{a}\sin 2\alpha}-\frac{\frac{H}{a}\frac{l}{a}\sin^2\alpha}{\left(1+\frac{l}{a}\sin\alpha\right)^2}\right\}$$

$$=\frac{m}{2}\omega^2l^2\lambda^2\left\{\cos\alpha-\frac{\frac{H}{a}\sin\alpha}{1+\frac{l}{a}\sin\alpha}\right\}^2+\frac{m}{2}k^2\omega^2\lambda^2\frac{\left(\frac{l}{a}\right)^2\sin^2\alpha}{\left(1+\frac{l}{a}\sin\alpha\right)^2}$$

$$\therefore\ \frac{l\omega^2}{g}=\frac{\dfrac{\left(2+\frac{2l}{a}\sin\alpha\cos\alpha\right)\left(1+\frac{l}{a}\sin\alpha\right)^2}{2\cos\alpha+\frac{l}{a}\sin 2\alpha}-\frac{H}{a}\frac{l}{a}\sin^2\alpha}{\left(\cos\alpha+\frac{l}{a}\sin\alpha\cos\alpha-\frac{H}{a}\sin\alpha\right)^2+\frac{k^2}{a^2}\sin^2\alpha}$$

For small values of α,

$$\frac{l\omega^2}{g}=1+\frac{2H}{a}\sin\alpha$$

As α increases, ω first increases, because the inclined hangers increase the restoring force. As α increases still more the frequency begins to decrease because the body is forced to rotate more and more during an oscillation. Very roughly the frequency is a maximum when $\tan\alpha$ is about $\frac{a}{H}$ and, again very roughly, this maximum is given by:

$$\frac{l\omega^2}{g}=\frac{1+\frac{2l}{H}}{\left(\frac{k}{H}\right)^2}\text{ approximately.}$$

Examples of the variation of $\frac{l\omega^2}{g}$ are given in Fig. 146.

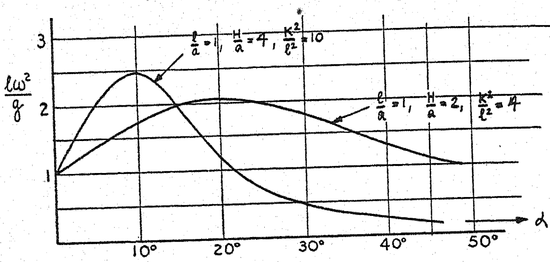


FIG. 146

Centerplate Spacing. The relation between the length of a car and the distance between the truck center plates has been the subject of much discussion and there is a divergence of opinion on this subject as on most others concerning car design.

The principal factors which must be considered in relation to riding and oscillation of the car are:

(*a*) There is one particular spacing of center plates for which a lateral disturbance at each truck is confined to that truck and does not affect the other truck. Therefore, it is not possible for a disturbance at one truck to be increased by anything coming from the other truck. This spacing must be such that the distance from the center of gravity of the body to each truck center plate is equal to the radius of gyration of the body about a vertical axis through the center of gravity. This will generally correspond to a center plate spacing about 60% of the length of the car.

(*b*) In order to guide the car into and out of curves as smoothly as possible, there is an advantage in putting the trucks as far apart as possible. They then have the greatest possible leverage with which to turn the body. While this is an advantage on curves it is correspondingly bad on rough spots on the track. The extreme case is that in which cars are articulated so that the ends of two adjacent cars rest on one truck center plate. In this case the truck spacing is, of course, equal to the length of the car.

In general neither of these considerations seems to be of much greater importance than the other and the majority of cars are built

with truck centers spaced roughly 75% of the length, this being determined more by practical design considerations than by a deliberate choice of centers from the riding quality viewpoint. There have probably been more instances of rough cars with short spacing than with long spacing and therefore the commonest preference is for long spacing, other things being equal.

Chapter 22

VIBRATION OF CAR BODIES

In most of the cases which we have considered, the car or locomotive body has been assumed to be rigid, so that it oscillates as one piece. Actually car bodies are comparatively flexible and we must, therefore, consider the vibrations which can occur in the body itself because of its flexibility.

We may distinguish between vibrations of some part of the body and vibrations of the whole body. In the first class are the familiar vibrations of panels, fittings, windows or other small parts, which vibrate by themselves, usually due to close synchronism of their natural frequencies with the frequency of some disturbance such as wheel revolutions, vibrations of suspension springs, etc. These vibrations are often almost undamped so that considerable vibration can be set up by a very small regular force. Such vibrations are often hard to predict, but easy to eliminate. The stiffness of the vibrating parts must be changed so as to avoid synchronism.

Vibrations of the entire body are more serious because they may be much harder to correct. Some car bodies, of composite construction with many imperfect joints, have sufficient internal friction to damp out vibration due to any small forces. If the body is welded and has few frictional contacts between parts it may have comparatively little internal friction and may easily pick up some small disturbance if the frequency is right. It may be necessary to use multiple sheets, rather lightly bolted together in those places in the body which bend the most, to provide some damping.

In other cases a car body may carry a partly unbalanced machine such as a Diesel engine or air compressor, which produces regular forced vibrations. The vibrations of car bodies are only slightly different from the many simple vibrations which we have already

studied. The difference is that while most of the simple systems can vibrate in only one or two ways, a flexible body can vibrate in any number of different ways. There are a series of "normal" ways in which the body vibrates which are illustrated in Fig. 147. For each of these cases the car can vibrate freely and every part of it moves in simple harmonic motion with the same frequency as every other part. In Fig. 147 (*a*) shows a 1-wave "normal" vibration, (*b*) a 2-wave shape, (*c*) a 3-wave shape, and there are any number of others, having more waves. The shapes with only a few waves in them have the lowest frequencies and are the most important. The importance of these "normal" vibrations comes from two sources:

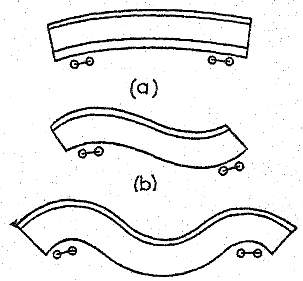


FIG. 147

(1) Any vibration can be split up into a series of "normal" vibrations taking place at once, and

(2) Each normal vibration consists of nothing but simple harmonic motion with one natural frequency. Thus each normal vibration is as easy to study as a simple weight on a spring.

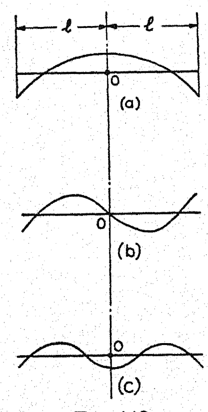


FIG. 148

Some of the most useful results can be seen at once by considering normal vibrations. In most cases the natural frequencies of vibration of a car body will be considerably higher than the natural frequency with which the body as a whole vibrates on the springs. This means that the springs act very nearly as constant upward forces which support the weight, but have only a minor effect on the normal vibrations of the body itself. This allows us to make some simple approximations which will give a good idea of the behavior of actual cars.

Vibrations of a Uniform Rod. This simple case will often serve to represent quite complicated structures. Fig. 148 shows the normal

1-, 2- and 3-wave vibrations of the rod, and there are any number of others with more waves.

It may be proved that the shapes can be represented by the following equations:

For 1 wave, as in Fig. 148 (*a*), or for 3 waves as in Fig. 148 (*c*), or for any other odd number of waves

$$y = y_0 \left[\frac{\cos \alpha l \cosh \alpha x + \cosh \alpha l \cos \alpha x}{\cos \alpha l + \cosh \alpha l} \right]$$

where $$\cos 2\alpha l \cosh 2\alpha l = 0$$

This equation has a series of roots,

$\alpha l = 0$, no waves, rigid body vibration
$\alpha l = 2.36$, one wave
$\alpha l = 3.93$, two waves
$\alpha l = 5.50$, three waves
$\alpha l = 7.07$, four waves
etc.

For an even number of waves, the shape is

$$y = \frac{y_0'}{\alpha} \left[\frac{\sin \alpha l \sinh \alpha x + \sinh \alpha l \sin \alpha x}{\sin \alpha l - \sinh \alpha l} \right]$$

In these equations

y_0 = the deflection at the center for an even number of waves
y_0' = the slope at the center for an odd number of waves
$2l$ = the length of the rod
x = distance along the rod, measured from the center
y = deflection of the rod at right angles to its length

The natural frequency of each normal vibration is given by

$$f = \frac{\alpha^2}{2\pi} \sqrt{\frac{EIg}{w}}$$

where α is obtained from the above equations,
E = modulus of elasticity of material, lb./sq. in.
I = moment of inertia of section of rod, in.^{4}

g = acceleration due to gravity, in./sec.^{2}
w = weight of rod in lb. per in. of length

It will be noticed that the values of αl are very nearly equal to

$$\alpha l = \left(\frac{2n + 1}{4}\right)\pi$$

where n = number of waves in the vibration.

Therefore, approximately, the frequency of an n-wave vibration is

$$f_n = \frac{(2n + 1)^2}{32} \frac{\pi}{l^2} \sqrt{\frac{EIg}{w}}$$

Example. A steel rod is 80 ft. long; it weighs 50 tons, evenly distributed; the moment of inertia of the section which resists bending is 8000 in.^{4}

Then $E = 30{,}000{,}000$ lb./in.^{2}
$I = 8{,}000$ in.^{4}
$g = 32.2 \times 12 = 386$ in./sec.^{2}
$l = \frac{80}{2} \times 12 = 480$ in.

$$w = \frac{50 \times 2000}{480} = 208\text{-lb./in.}$$

and

$$f_1 = \frac{3^2}{32} \times \frac{3.14}{480^2} \sqrt{\frac{30{,}000{,}000 \times 8{,}000 \times 386}{208}}$$

$= 2.55$ cycles per sec., fundamental frequency.

The 2-wave frequency is approximately $\frac{25}{9} \times 2.55 = 7.10$ cycles per sec., and so forth.

This corresponds roughly to a car body.

Forced Vibrations. For each normal type of vibration there are a certain number of points in the beam which do not move. For a single wave, there are two points, for a double wave, three points and so on. These points are called nodes. Take, for example, a 1-wave vibration produced by an oscillating force. If the force acts at the middle, it will vibrate the beam up and down, but if it acts at one of the nodes, which remains stationary, it cannot produce any 1-wave

vibration, although it will probably produce 2- and 3-wave vibrations. In general no vibration can be forced which has a node or stationary point at the place where the force acts.

This is of importance in choosing where to locate an unbalanced machine on a car body. For example, if we have a car body similar to that in the example above whose fundamental frequency is 2.55 cycles per sec., and a reciprocating compressor operating at 420 r.p.m., this will be almost exactly in synchronism with the 2-wave frequency of 7.10 per sec. or 425 per min. If the compressor can be located at one of the nodes of the 2-wave vibration, for example in the middle of the car, it will not be likely to cause much vibration. If, however, the compressor is set off the center, it will tend to set up a resonant 2-wave vibration which may be very severe.

Cars may carry internal combustion engines which are balanced so far as forces are concerned, but which have unbalanced moments which act on the supports. Such an engine will only produce vibrations which result in the engine itself vibrating. In Fig. 149 (*a*) shows an engine with a fore-and-aft unbalanced moment placed in the center of a car. There will be no odd-wave vibrations, but the 2- and 4-wave motions probably will be most noticeable. In Fig. 149 (*b*) the same engine, located towards the end of the car, will tend to produce a considerable 1-wave vibration.

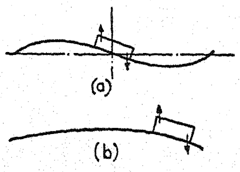


FIG. 149

Example. This simple case will illustrate an approximate method of calculating the amount of forced oscillations. We will take a uniform car body so as to avoid lengthy calculations of deflections, etc., but the method can easily be applied to actual vehicles with variable weight and stiffness. Suppose that on the 80-ft. car referred to in the example above we have an engine which is balanced except for the secondary unbalance of a reciprocating compressor. The engine weighs 10,000 lb. and the unbalance is equivalent to a weight of 2 lb. moving with a stroke of 10 in. at twice the engine r.p.m. The engine speed is variable with a maximum of 600 r.p.m. The engine is located 10 ft. from the center of the car.

(*a*) Rigid body vibration. The car weighs 100,000 lb. and the

engine 10,000 lb. The center of gravity is 0.91 ft. from the center. The movement of 2 lb. with a stroke of 10 in. will make the center of gravity move with a total amplitude of

$$\frac{10 \times 2}{110{,}000} = 0.18 \text{ mil.}$$

The force acts 10 ft. from the center, the radius of gyration of the car about its center of gravity is k ft., where

$$k^2 = \frac{80^2}{12} = (23.1)^2$$

Therefore the center of oscillation of the car body is

$$\frac{k^2}{10} \text{ ft. from the center.}$$

$$= \frac{80^2}{12 \times 10} = 53.4 \text{ ft. from the center.}$$

This point does not move. The center of gravity is $53.4 + 0.91 =$

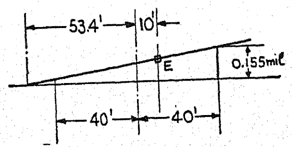


FIG. 150

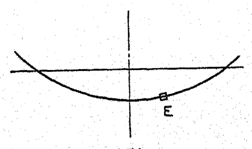


FIG. 151

54.31 ft. from this point. (Fig. 150). The maximum total amplitude at the end of the car is $0.18 \text{ mil.} \times \dfrac{93.4}{54.31} = 0.31$ mil. The semi-amplitude is 0.155 mil.

(*b*) One-wave vibration. In the formula given above for a 1-wave vibration, it will be seen that $\cosh \alpha l$ is large compared to $\cos \alpha l$ and, therefore, approximately,

$$y = y_0 \cos \alpha x = y_0 \cos \frac{2.36x}{l}$$

or

$$y = y_0 \cos \frac{2.36x}{40}$$

If this is supposed to be produced by the gradual application of a unit force at E (Fig. 151), where the engine is located, the work done by the unit force is

$$\tfrac{1}{2} \times \text{unit force} \times \text{deflection at } E.$$

$$= \frac{1}{2} y_0 \cos \frac{2.36 \times 10}{40} = \frac{1}{2} y_0 \cos 0.59$$

$$= \tfrac{1}{2} y_0 \cos 34^\circ = 0.41\, y_0$$

This must equal the energy stored in bending the car body which is

$$\frac{1}{2} \int_{-l}^{+l} EI \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{EI}{2} y_0^2 \times \left(\frac{2.36}{l}\right)^4 \int_{-l}^{+l} \cos^2 \frac{2.36x}{l}\, dx$$

$$= \frac{EI}{2} y_0^2 \left(\frac{2.36}{l}\right)^3 \left[\frac{2.36x}{2l} + \frac{1}{4} \sin \frac{2 \times 2.36x}{l}\right]_{-l}^{+l}$$

$$= \frac{EI}{2} y_0^2 \left(\frac{2.36}{l}\right)^3 [2.36 + \tfrac{1}{2} \sin 4.72]$$

$$= \frac{EI}{2} y_0^2 \left(\frac{2.36}{480}\right)^3 [2.36 + 0.50]$$

Therefore,

$$0.41\, y_0 = \frac{30 \times 10^6 \times 8000}{2} y_0^2 \times \left(\frac{2.36}{480}\right)^3 \times 2.86$$

or

$$y_0 = \frac{0.41 \times 2}{240{,}000 \times 2.86} \left(\frac{4.80}{2.36}\right)^3$$

$$= .01 \text{ mil. per unit force}$$

At an engine speed of n r.p.m. the maximum force due to the unbalance is $\frac{w}{g} r\omega^2$

$$= \frac{2}{32.2} \times \frac{5}{12} \times \left(\frac{4\pi n}{60}\right)^2$$

$$= 4.1 \times \left(\frac{n}{60}\right)^2$$

which reaches a maximum of 410 lb. at 600 r.p.m. Also the natural frequency of 1-wave vibration (neglecting the effect of the weight of the engine) was calculated above as 2.55 cycles per sec.

Therefore, the semi-amplitude of the 1-wave vibration, measured at the center, is

$$\frac{.012 \times 4.1 \left(\frac{n}{60}\right)^2}{1 - \left(\frac{n}{60 \times 2.55}\right)^2} \text{ in.}$$

At 600 r.p.m. this is

$$\frac{.012 \times 4.1 \times 100}{1 - \left(\frac{10}{2.55}\right)^2} = -\ 0.34 \text{ in.}$$

There is 1-wave resonance at the natural frequency of 153 per min., which corresponds to 76.5 r.p.m. of the engine.

The 2-wave, 3-wave and other vibrations are similarly calculated and the resultant motion is the sum of all the normal modes, calculated separately. Resonance of different modes of vibrations will be approximately at the following engine speeds.

1-wave		76.5 r.p.m.
2-wave	$76.5 \times \frac{25}{9} =$	212 r.p.m.
3-wave	$76.5 \times \frac{49}{9} =$	416 r.p.m.
4-wave	$76.5 \times \frac{81}{9} =$	688 r.p.m.

The unbalance is located near a node of the 3-wave vibration, but nearly half-way between nodes of the 4-wave vibration, so that a 4-wave vibration is particularly to be feared at maximum engine speed.

More Accurate Calculation. The section above indicates rather briefly the method of splitting up a vibration into its various "normal modes" and considering each one separately. This is valuable because it helps in seeing what is taking place in an apparently complicated vibration. Various refinements can also be made in the methods of calculation if greater accuracy is necessary. However, where a really accurate calculation of a real problem is necessary, a

different method, which will be briefly indicated, is preferable. Space allows only the general principle of the method to be given. The details are not difficult to work out.

In general the vibrating body has a variable distribution of weight and variable stiffness. The first object is to find the shape of the vibrating body at the extreme point of the vibration. We assume a suitable frequency f. Then if a weight w has a semi-amplitude of vibration y, it exerts a maximum force on the car body of $\frac{w}{g}\,\omega^2 y$.

The body must have a shape corresponding to no bending moment or shearing force at either of the free ends. First take the case where at one end, say the left, the deflection is unity and the slope zero, with both bending moment and shear zero. Then we can construct the shape of the body by starting at the left and proceeding step by step, as though calculating the deflection of a beam under a series of loads. The load due to each weight w is $\frac{w}{g}\,\omega^2 y$. Since the loads depend on the deflections, they are calculated step by step as the deflection curve is obtained. Finally we arrive at the right-hand end of the body with a bending moment M_1 and a shear F_1.

Now we repeat the process, but start with zero deflection and unit slope, finally ending up at the right-hand end with a bending moment M_2 and shear F_2.

Now if we take the first calculation and multiply all the deflections, forces, etc., by M_2, then multiply the second calculation by M_1 and subtract one from the other we obtain a composite result in which, at the right-hand end of the body, the bending moment is $M_1 \times M_2 - M_2 \times M_1 = 0$ and the shear is $F_1 \times M_2 - F_2 \times M_1$. If we have chosen the frequency f so that it is one of the natural frequencies of normal vibration it will be found that the resultant shear is zero and we have constructed the shape of the vibration. If the shear is not zero we take another frequency and repeat the calculation. Usually two or at most three calculations will give the natural frequency and the shape accurately. The approximate methods given previously are used to obtain a fairly close estimate of the frequency before starting the detail calculation.

The method may be extended to cover a forced oscillation in the

following way. Here the frequency of the disturbing force will be known. We start at the left end as before, first with unit deflection and zero slope, then with zero deflection and unit slope. We arrive at the point where the disturbing force acts, having calculated M_1 and F_1 in the first case and also the deflection y_1 and the slope s_1. In the second case we calculate M_2, F_2, y_2 and s_2. We now start at the right-hand end and make similar calculations, arriving at the disturbing force with $M_1{}^1$, $F_1{}^1$, $y_1{}^1$, $s_1{}^1$ and $M_2{}^1$, $F_2{}^1$, $y_2{}^1$, $s_2{}^1$. The final composite curve is

$$(k_1 \times \text{curve } 1) + (k_2 \times \text{curve } 2) + (k_3 \times \text{curve } 1^1) + (k_4 \times \text{curve } 2^1)$$

The four constants $k_1 \cdots k_4$ are found from the four relations, y and s are continuous through the point of application of the disturbing force, and the difference in moment and shear on the two sides of the force equals the disturbing moment and force.

The calculations are rather laborious but are not difficult if set up systematically and short cuts will become evident in most cases. In actual practice critical speeds can be predicted accurately by these methods even for structures of very variable stiffness, such as Diesel locomotives.

Chapter 23

RAIL VIBRATION

A rail acts as a spring support for the wheels which run on it and the wheels can vibrate on the rail as they would vibrate on any other spring.

In general, serious rail vibrations are too fast to be transmitted through the vehicle suspension springs and therefore they have little or no effect on the spring-supported body of the vehicle.

In studying the rail vibrations, therefore, it is best to start by considering the sprung body as moving steadily forward, while the unsprung wheels, etc., vibrate independently on the rail.

The most interesting case of rail vibration is that which occurs under the driving wheels of a reciprocating steam locomotive at high speed.

Due to the partial balancing of the reciprocating parts, the drivers are out of balance and vibrate up and down on the rail under the effect of the unbalanced force, which increases with the square of the speed. Because the rail is a spring, the vibration increases as the frequency approaches the resonant frequency of vibration of the unsprung mass on the rail.

If the speed is high enough, which may be when the drivers slip, the vibration may be severe enough to lift the wheels off the rail during part of their revolution. When the wheel is supported by the rail during only part of the time, it has the effect of reducing the average stiffness of the support and thereby reducing the resonant frequency. Thus, when the wheels leave the rail, the resonant frequency drops and the vibration increases. The greater the vibration, the less the resonant frequency, so that the vibration will increase until it may reach or pass the point of resonance.

Below resonance the wheel will vibrate up and down with the overbalance weight. Above resonance the phase reverses, as in all similar

vibrations, and the wheel will be up when the unbalance is down. These effects have been observed during the slipping of high speed locomotives.

Example. In a steam locomotive:

w = unsprung weight per driving wheel = 6500 lb.
D = driving wheel diameter = 72 in.
n = overbalance weight at the crank pin = 200 lb.
r = crank radius = 18 in.
total wheel load at the rail = 33,000 lb.
W = spring-borne load per wheel = 33,000 − 6,500 = 26,500 lb.
V = speed at which the drivers are going, m.p.h.

For the rail we assume:

Weight per yard = 110 lb.
E = modulus of elasticity of steel = 30,000,000-lb./in.^{2}
I = moment of inertia of rail section = 55 in.^{4}
U = modulus of resistance of track foundation = 2000 lb./in.^{2}

Then the flexibility of the rail under a load is, from equation (43) below,

$$\frac{1}{2 \times \sqrt[4]{4EIU^3}} = \frac{1}{2 \times \sqrt[4]{4 \times 30 \times 10^6 \times 55 \times 2000^3}} = \frac{1}{170{,}500} \text{ in./lb.}$$

The deflection of the rail under the unsprung weight above is

$$d = \frac{6500}{170{,}500} = .038 \text{ in.}$$

The natural frequency of oscillation of the unsprung weight (neglecting the effect of the suspension spring) is

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{g \times 12}{d}} = \frac{1}{6.28}\sqrt{\frac{32.2 \times 12}{.038}} = 16.1 \text{ cycles per sec.}$$

The frequency of rotation of the drivers is

$$f = \frac{V \times \left(\frac{5280}{3600}\right)}{\left(\frac{\pi D}{12}\right)} = V \times \frac{5280}{3600} \times \frac{12}{3.14 \times 72} = .078V \text{ cycles per sec.}$$

The force due to overbalance is

$$\left(\frac{n \times 2r}{D}\right) \times \frac{\left(\frac{5280}{3600} V\right)^2}{\left(\frac{D}{24}\right) \cdot g} = 3.2 \frac{nrV^2}{D^2} = \frac{3.2 \times 200 \times 18}{72^2} V^2$$

$$= 2.2V^2 \text{ lb.}$$

The "dynamic magnifier" due to the spring support is

$$\frac{1}{1 - \left(\frac{f}{f_0}\right)^2} = \frac{1}{1 - \left(\frac{.078V}{16.1}\right)^2}$$

The total force at the rail due to the unbalance is therefore

$$\frac{2.2V^2}{1 - \left(\frac{.078V}{16.1}\right)^2}$$

This force is plotted in Fig. 152 and it will be seen that at 105 m.p.h. the force becomes equal to the static wheel load, so that the net load on the rail fluctuates from zero to twice the static load. At higher speeds the wheel leaves the rail during part of its revolution. When the wheel is not in contact with the rail, it is still acted on by the locomotive springs. Let the stiffness of the system resisting movement of one driving wheel be

$$13{,}000 \text{ lb./in.}$$

Then the deflection of this system due to the unsprung weight of one wheel is

$$d_1 = \frac{6{,}500}{13{,}000} = 0.5 \text{ in.}$$

The natural frequency of oscillation of the unsprung weight on the locomotive spring system alone is

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{g \times 12}{d_1}} = \frac{1}{6.28}\sqrt{\frac{32.2 \times 12}{0.5}} = 4.4 \text{ cycles per sec.}$$

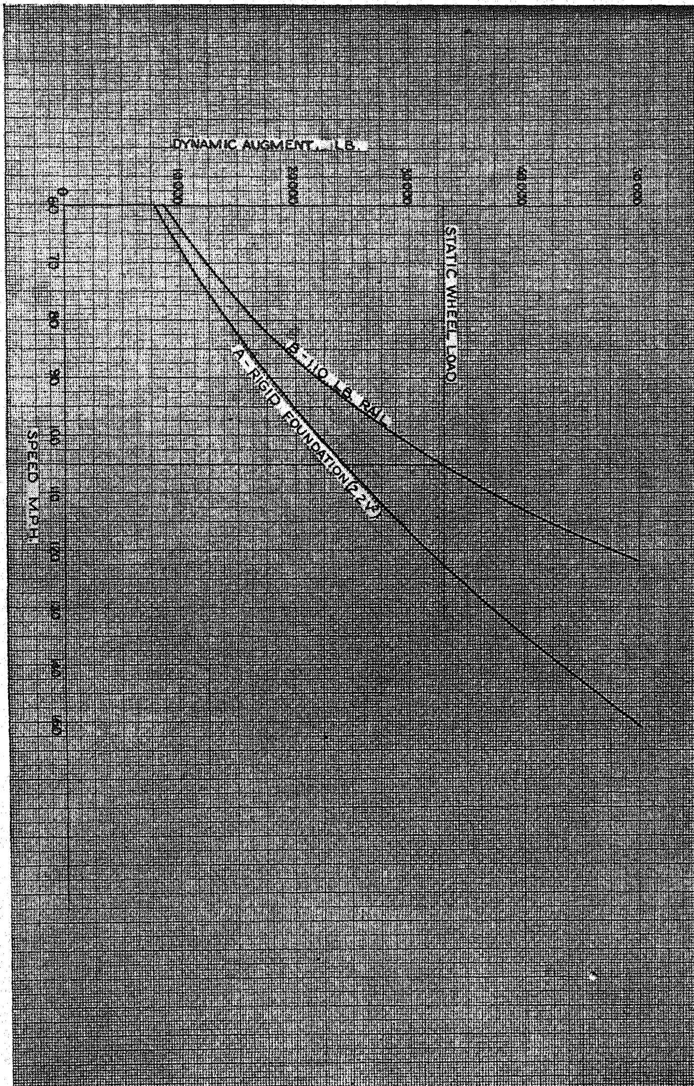


Fig. 152. Rail Vibration

The extreme case of nearly resonant vibration is approximately as shown in Fig. 153. The wheel is in contact with the rail during a time $\frac{1}{2f_0} = \frac{1}{2 \times 16.1} = .031$ sec. It is in the air during a time $\frac{1}{2f_1} = \frac{1}{2 \times 4.4} = 0.114$ sec. Hence the total period is $0.031 + 0.114 = 0.145$ sec. and the frequency is $\frac{1}{0.145} = 6.9$ cycles per sec.

Since the frequency of rotation is $0.078\ V$, the speed at which resonance can occur is approximately $V = \frac{6.9}{.078} = 88.5$ m.p.h.

It will be seen that if the speed of revolution of the drivers exceeds 105 m.p.h., the wheels will leave the rail. The natural frequency will then decrease, the wheels will rise more until finally a fairly steady state is reached in which the vibration corresponds to a natural frequency at about 88.5 m.p.h. The wheel will then be vibrating above its natural frequency and the motion will be opposite in phase to the overbalance weight, that is, the wheel will be up when the weight is down.

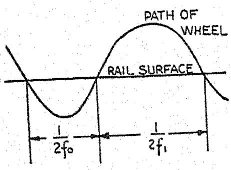


FIG. 153

More detailed studies of this and other problems of rail vibration are considered in the following sections in which more mathematic knowledge is assumed.

Deflection of Rail. We assume that the rail is uniform and uniformly supported on an elastic foundation.

Let E = modulus of elasticity of rail steel, lb./sq. in.

I = moment of inertia of section of rail for bending in a vertical plane, in.^{4}

u = modulus of resistance of the foundation, lb. per in. deflection per in. of length.

P = wheel load on the rail, lb.

$$\alpha = \sqrt[4]{\frac{u}{4EI}}$$

M = bending moment in the rail, lb.-in.

F = shearing force in the rail, lb.
x = distance along the rail, in.
y = downward deflection of the rail, in.

A small section of the rail, seen from the side, is shown in Fig. 154. Its length is δx and the forces acting on it are shown in the diagram. The forces must be in equilibrium, therefore, equating the total forces and moments to zero,

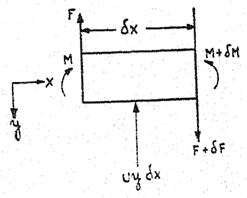


FIG. 154

$$\delta F = uydx \quad \text{or} \quad \frac{dF}{dx} = uy$$

and
$$\delta M = F\,\delta x \quad \text{or} \quad \frac{dM}{dx} = F$$

Also, from the simple theory of bending,

$$M = -EI\frac{d^2y}{dx^2}$$

Therefore, differentiating twice,

$$-EI\frac{d^4y}{dx^4} = \frac{d^2M}{dx^2} = \frac{d}{dx}\left(\frac{dM}{dx}\right) = \frac{dF}{dx} = uy$$

or
$$\frac{d^4y}{dx^4} = -\frac{u}{EI}y = -4\alpha^4y$$

or
$$\frac{d^4y}{dx^4} + 4\alpha^4y = 0 \quad \ldots\ldots\ldots\ldots \quad (42)$$

The solution of this equation is

$$y = e^{\alpha x}(A\cos\alpha x + B\sin\alpha x) + e^{-\alpha x}(C\cos\alpha x + D\sin\alpha x)$$

where A, B, C, D are constants.

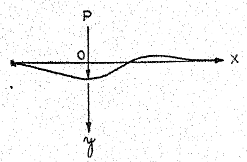


FIG. 155

Fig. 155 shows a single load P on the rail, at the origin of coordinates.

Then, considering the section of the rail to the right of the load, we have:

When x is large, y is zero, therefore A and B are zero.

When $x = 0$, the slope is zero, by symmetry, hence $0 = -C + D$

and
$$y = Ce^{-\alpha x}(\cos\alpha x + \sin\alpha x)$$

To find C we use the condition that the foundation under this half of the rail must carry half the total load.

Therefore $$\frac{P}{2} = \int_0^\infty uy\,dx$$

$$= -\frac{Cu}{\alpha}[e^{-\alpha} \cos \alpha x]_0^\infty$$

$$= \frac{Cu}{\alpha}$$

Therefore $$C = \frac{P\alpha}{2u}$$

and $$y = \frac{P\alpha}{2u} e^{-\alpha x} (\cos \alpha x + \sin \alpha x)$$

The maximum deflection, under the load, where $x = 0$, is

$$y_0 = \frac{P\alpha}{2u} = \frac{P}{2\sqrt[4]{4EIu^3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

The maximum bending moment, under the load, is

$$M_0 = -EI\frac{d^2y}{dx^2} = -EI\left(-2\alpha^2\frac{P\alpha}{2u}\right) = \frac{P\alpha^3 EI}{u} = \frac{P}{4\alpha}$$

The bending moment curve is

$$M = \frac{P}{4\alpha} e^{-\alpha x}(\cos \alpha x - \sin \alpha x)$$

The distance from the load to the nearest point at which the bending moment is zero is given by

$$O = \frac{P}{4\alpha} e^{-\alpha x} (\cos \alpha x_1 - \sin \alpha x_1)$$

$$\therefore \tan \alpha x_1 = 1$$

$$\therefore \alpha x_1 = \frac{\pi}{4}$$

$$\therefore x_1 = \frac{\pi}{4\alpha} = \frac{\pi}{4}\sqrt[4]{\frac{4EI}{u}}$$

Curves of deflection and bending moment against distance are given in Fig. 156.

Oscillating Load on a Rail. The rail acts as a spring. A weight placed on the rail can oscillate as if it were on any other kind of spring. As in other cases, the mass of the spring itself is often negligible when

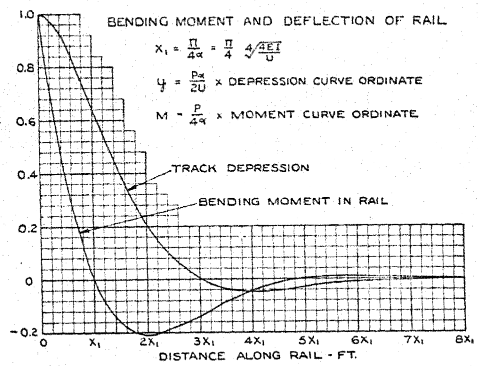


FIG. 156

the supported weight is large and the frequency of oscillation is not too great.

A weight W placed on a rail will deflect the rail

$$y_0 = \frac{W\alpha}{2u}$$

Therefore, the natural frequency of oscillation is

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{2ug}{W\alpha}}$$

If the weight is acted on by an oscillating force

$$F = F_0 \cos 2\pi ft$$

the forced oscillation will be

$$y = (F_0 \cos 2\pi ft)\left(\frac{\alpha}{2u}\right)\left(\frac{1}{1 - \dfrac{f^2}{{f_0}^2}}\right)$$

We can approximate the effect of the mass of the rail by considering that every section of rail, in its extreme position, is acted on by a force $uydx$ from the foundation and also an inertia force $-\frac{wy}{g}(2\pi f)^2\,\delta x$ where w is the weight of rail per in.

Hence the mass of the rail has the same effect as changing the modulus of resistance of the foundation from u to $\left(u - 4\pi^2 f^2 \frac{w}{g}\right)$.

Typical values for a locomotive driving wheel on 130-lb. rail at 90 m.p.h. are:

$$u = 2000\text{-lb./in.}^2$$

$$f = 6 \text{ rev. per sec.}$$

$$w = \tfrac{130}{36} = 3.6 \text{ lb./in.}$$

$$g = 386 \text{ in./sec.}^2$$

Therefore, $4\pi^2 f^2 \frac{w}{g} = 13.2$ lb./in.^{2} which is a correction to u of only 0.66% and quite negligible.

Moving Load on a Rail. At low speeds there is little difference between the deflection of a rail under a stationary or a moving load, the chief effect being due to friction in the roadbed which resists the deflection of the rail and causes "rolling friction." At very high speeds, however, the inertia of the rail becomes noticeable. The forces acting on a small element when the rail is moving are shown in Fig. 157. The forces are no longer in equilibrium, however, and we

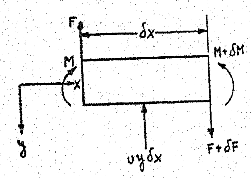


FIG. 157

have, for this element, whose mass is $\frac{w}{g}\,\delta x$ the equation,

$$\left(\frac{w}{g}\,\delta x\right)\frac{\partial y^2}{\partial t^2} = \delta F - uy\,\delta x$$

Hence
$$\frac{\partial F}{\partial x} = uy + \frac{w}{g}\frac{\partial^2 y}{\partial t^2}$$

and instead of equation (42) we have

$$\frac{\partial^4 y}{\partial x^4} + \frac{w}{gEI}\frac{\partial^2 y}{\partial t^2} + 4\alpha^4 y = 0 \quad . \quad . \quad . \quad . \quad . \quad (44)$$

If the load is steady and progressing along the rail with uniform velocity v, we have a rail deflection which also moves uniformly along the rail with velocity v, that is the deflection is a function of $(x - vt)$ and we can write $y = f(x - vt)$. Substituting in equation (44), we have, writing D for $\dfrac{\partial}{\partial(x - vt)}$

$$\left(D^4 + \frac{wv^2}{gEI}D^2 + 4\alpha^4\right)f(x - vt) = 0$$

As long as $\dfrac{wv^2}{gEI}$ is less than $4\alpha^2$ this equation has a solution

$$f(x - vt) = e^{p(x-vt)}\left[A\cos q(x - vt) + B\sin q(x - vt)\right]$$
$$+ e^{-p(x-vt)}\left[C\cos q(x - vt) + D\sin q(x - vt)\right]$$

As before, when $x - vt$ is positive,

$$A = B = 0 \quad \text{and} \quad 0 = -pC + qD$$

Therefore

$$f(x - vt) = \frac{Ce^{-p(x-vt)}}{q}\left[q\cos q(x - vt) + p\sin q(x - vt)\right]. \quad (45)$$

and

$$\frac{P}{2} = \frac{2pC}{p^2 + q^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Calculating p and q from the roots of the equation

$$D^4 + \frac{wv^2}{gEI}D^2 + 4\alpha^4 = 0$$

we find

$$p^2 + q^2 = 2\alpha^2$$

and $p = \sqrt{2}\,\alpha\cos\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right)$ where $\theta = \sin^{-1}\left(\dfrac{wv^2}{4gEI\alpha^2}\right)$.

These relations are readily obtained by sketching the roots on an Argand diagram.

From (45) and (46) the deflection under the load is

$$\frac{P(p^2 + q^2)}{4p} = \frac{P\alpha}{2\sqrt{2}\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right)}.$$

When $\theta = \frac{\pi}{2}$ the denominator becomes zero and the deflection is infinite, indicating resonance between the speed of the load and the speed of the wave in the rail. This critical speed is given by

$$\frac{wv^2}{4gEI\alpha^2} = \sin\frac{\pi}{2} = 1$$

or

$$v^2 = \frac{4gEI\alpha^2}{w} = \frac{2g}{w}\sqrt{EIu} \qquad (47)$$

Oscillating Wheel Leaving the Rail. When a moving wheel acted on by an oscillating force leaves the surface of the rail during part of its oscillation, the motion is made up of two parts which can be calculated separately and pieced together to represent the complete oscillation.

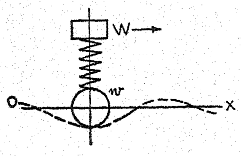


FIG. 158

In Fig. 158 we take OX as the normal unloaded surface of the rail. The weight W is assumed to move steadily forward, unaffected by the oscillations of the wheel, at a height corresponding to its height at rest with normal spring and rail deflections. The wheel, whose weight is w, is acted on by an alternating force $F \cos \omega t$.

Let k_s = stiffness of the spring
k_r = stiffness of the rail acting as a spring support.

Then for vibrations in contact with the rail, the wheel has a frequency f_0 corresponding to

$$\omega_0 = 2\pi f_0 = \sqrt{\frac{g(k_r + k_s)}{w}}.$$

For vibrations not in contact with the rail, the natural frequency is 1, corresponding to

$$\omega_1 = 2\pi f_1 = \sqrt{\frac{gk_s}{w}}.$$

Now take half a complete oscillation, as in Fig. 159, and suppose that we start at A, the lowest point of the path where the deflection is y_0. After a certain time T we arrive at B where the wheel leaves the rail and jumps to C, which is the highest point and is a distance y_1 above the rail.

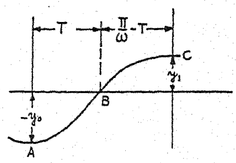


FIG. 159

Measuring y upwards, we have for the section AB,

$$\frac{w}{g}\frac{d^2y}{dt^2} + (k_r + k_s)\left(y + \frac{W + w}{k_r}\right) + F \cos \omega t = 0$$

from which

$$y = \left[y_0 + \frac{(W + w)g}{w(\omega_0^2 - \omega_1^2)} + \frac{Fg}{w(\omega_0^2 - \omega^2)}\right] \cos \omega_0 t - \frac{(W + w)g}{w(\omega_0^2 - \omega_1^2)} - \frac{Fg \cos \omega t}{w(\omega_0^2 - \omega^2)}$$

For the section BC, similarly,

$$y = \left[y_1 + \frac{(W + w)g\omega_0^2}{w\omega_1^2(\omega_0^2 - \omega_1^2)} - \frac{Fg}{w(\omega_1^2 - \omega^2)}\right] \cos \omega_1 \left(\frac{\pi}{\omega} - t\right) - \frac{(W + w)g\omega_0^2}{w\omega_1^2(\omega_0^2 - \omega_1^2)} - \frac{Fg \cos \omega t}{w(\omega_1^2 - \omega^2)}$$

If T is the time at which $y = 0$ and the slope of the path is continuous at this time, we have the above two equations equal to zero when $t = T$ and also

$$-\left[\frac{(W + w)g}{w(\omega_0^2 - \omega_1^2)} + \frac{Fg \cos \omega T}{w(\omega_0^2 - \omega^2)}\right] \omega_0 \tan \omega_0 T + \frac{\omega Fg \sin \omega T}{w(\omega_0^2 - \omega^2)}$$

$$= \left[\frac{(W+w)g\omega_0^2}{w\omega_1^2(\omega_0^2-\omega_1^2)} + \frac{Fg \cos \omega T}{w(\omega_1^2-\omega^2)}\right] \omega_1 \tan \omega_1 T + \frac{\omega Fg \sin \omega T}{w(\omega_1^2-\omega^2)} \quad (48)$$

This equation gives T in terms of the known quantities. It is most easily handled by taking a series of values of T from 0 to $\frac{\pi}{\omega}$ and calculating the corresponding values of F. From this curve between T and F we can pick off the correct value of T, after which y_0 and y_1 can be calculated if required.

As an example, we will consider a case in which the wheel spends half its time in contact with the rail and the other half in the air, that is $T = \frac{\pi}{2\omega}$.

Substituting this value in (48) we obtain,

$$\frac{\omega F}{(W + w)}\left[\frac{\omega_1^2 - \omega_0^2}{(\omega_0^2 - \omega^2)(\omega_1^2 - \omega^2)}\right] = \frac{\omega_1\omega_0}{\omega_1^2\,(\omega_0^2 - \omega_1^2)}\left[\omega_0 \tan\frac{\pi}{2}\frac{\omega_1}{\omega} + \omega_1 \tan\frac{\pi}{2}\frac{\omega_0}{\omega}\right]$$

The interesting case is that in which ω lies between ω_0 and ω_1. Then in the above equation,

$$F \times \text{a positive quantity} = \frac{2\omega}{\pi\omega_1}\tan\frac{\pi\omega_1}{2\omega} + \frac{2\omega}{\pi\omega_0}\tan\frac{\pi\omega_0}{2\omega}. \qquad (49)$$

To examine the right-hand side of this equation, we sketch the graph of $y = \frac{1}{\theta}\tan\theta$ in Fig. 160. We can pick off equal ordinates such as PQ and P^1Q^1 and construct the curve in Fig. 161 of values of $\frac{\omega_1}{\omega}$ and $\frac{\omega_0}{\omega}$ for which

$$\frac{2\omega}{\pi\omega_1}\tan\frac{\pi\omega_1}{2\omega} + \frac{2\omega}{\pi\omega_0}\tan\frac{\pi\omega_0}{2\omega} = 0$$

We know ω_0 and ω_1 so we can draw OR representing the ratio of

$$\frac{\left(\frac{\omega_0}{\omega}\right)}{\left(\frac{\omega_1}{\omega}\right)} = \frac{\omega_0}{\omega_1}$$

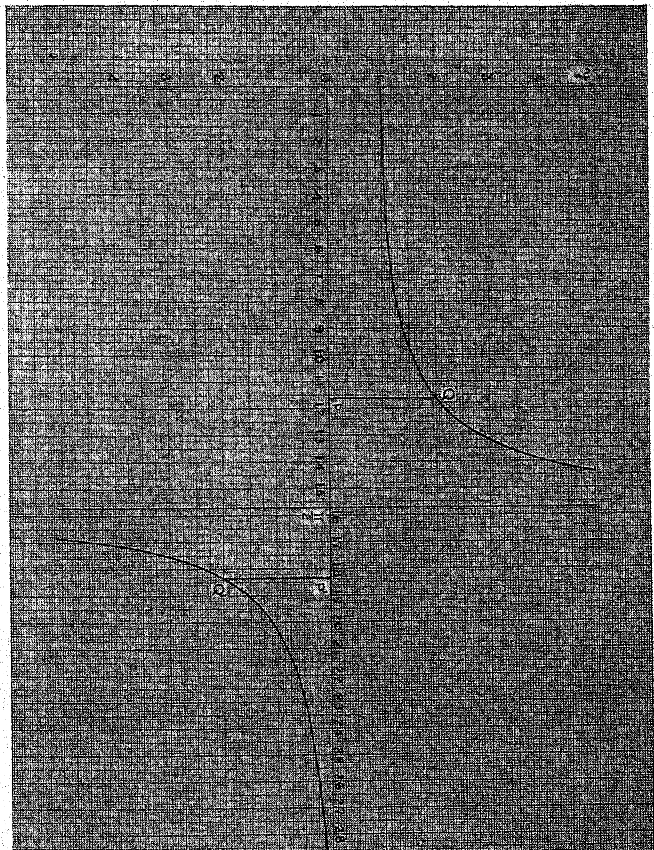


FIG. 160. RAIL VIBRATION

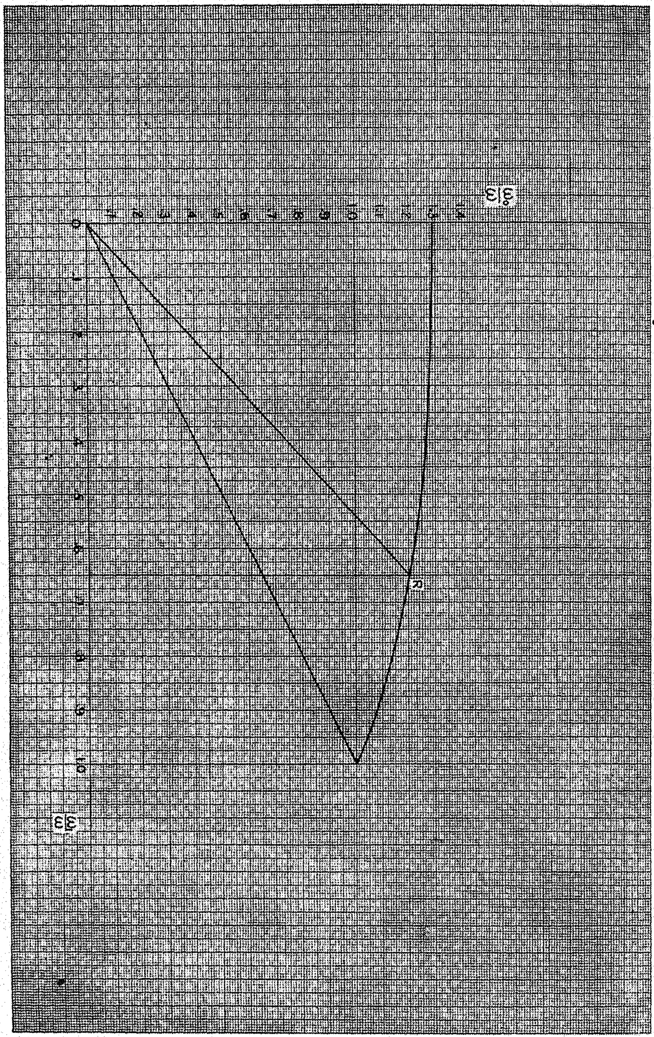


FIG. 161. RAIL VIBRATION

Then the point R gives a value of ω for which $F = 0$ which means a finite vibration resulting from an infinitesimal force, or resonance.

Now if P and P^1 in Fig. 160 represent $\frac{\pi\omega_0}{2\omega}$ and $\frac{\pi\omega_1}{2\omega}$ for the critical value of ω it will be seen that if ω increases, P and P^1 move to the left and the right-hand side of equation (49) becomes negative, so that F is opposite in phase to the motion.

Actually, of course, if ω changes, T will change and the condition of resonance will change, but the reasoning above is sufficient to show the existence of resonance and the reversal of phase when the forced frequency is higher than the resonant frequency.

Impact with the Rail. After a wheel has left the rail due to vertical vibration, it passes through a short air path and then returns to contact with the rail with an impact. In order to get an approximate idea of the magnitude of this impact, we will calculate it assuming that waves travel infinitely fast through the rail. An impact is so short that the rail has no time to deflect, it merely acquires a downward velocity and hence the foundation forces, which are brought into play by rail deflection, will not act during the impact. If these forces are neglected, the equation of the rail becomes

$$\frac{\partial^4 y}{\partial x^4} + \frac{w}{gEI}\frac{\partial^2 y}{\partial t^2} = 0$$

and if we assume a solution of the form

$$y = f(x) \sin \omega t$$

$$\left(D^4 - \frac{w\omega^2}{gEI}\right) f(x) = 0$$

For positive values of x, if the rail is to remain undisturbed at infinity,

$$y = Ae^{-px} \sin \omega t$$

where

$$p = \sqrt[4]{\frac{w\omega^2}{gEI}}$$

The initial velocity of the rail at the point of impact is

$$\left(\frac{dy}{dt}\right)_0 = A\,\omega$$

also the total initial momentum of the rail is

$$2\int_0^\infty \frac{w}{g}\left(\frac{dy}{dt}\right) dx = (A\,\omega)\left(\frac{2w}{gp}\right)$$

so that the entire rail acts at the moment of impact as if it were a rigid piece of length $\frac{2}{p}$.

Assuming that ω is approximately the angular velocity corresponding to the vibration in contact with the rail,

$$\omega^2 = \frac{2ug}{W\alpha}$$

where W is the weight of the wheel

Then $$p^4 = \frac{w\omega^2}{gEI} = \frac{8w}{W}\alpha^3$$

and the equivalent rigid length of rail is

$$\frac{2}{p} = \sqrt[4]{\frac{2W}{w\alpha^3}}$$

It was found above that an approximation to the shape of the rail at impact is

$$y = Ae^{-px}\sin\omega t \quad (x \text{ positive})$$

This gives the shape shown in Fig. 162 with a kink at the point of impact. The theory is, of course, not exact enough to determine the shape very accurately, but it is a good indication of the tendency, observed in practice, of rails to be kinked by wheels which leave the rail and return to it with impact.

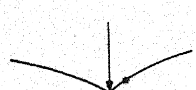


FIG. 162

Fig. 163 shows an actual record of the vertical movements of the driving wheels of a steam locomotive, when, due to slipping, they were

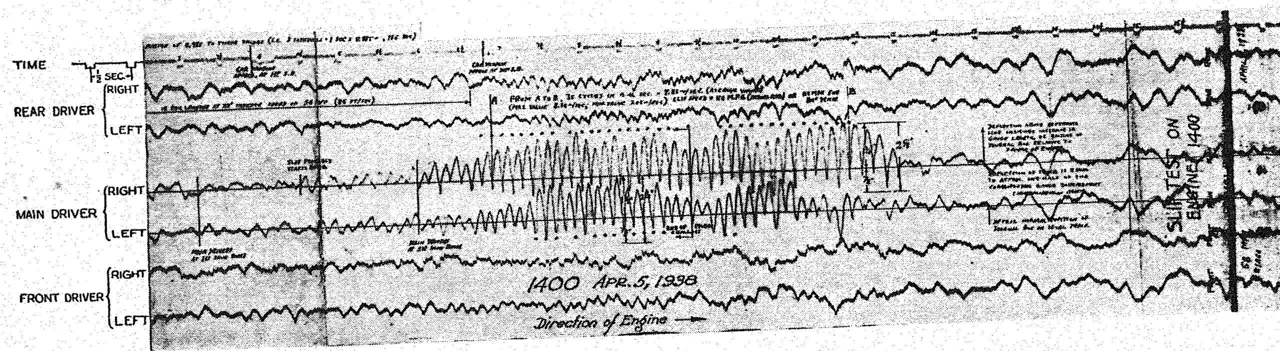


FIG. 163. VERTICAL MOVEMENT OF 4–6–2 STEAM LOCOMOTIVE DRIVERS
Locomotive speed: 58 m.p.h. Average driver speed, slipping on greased rail: 110 m.p.h.

revolving at excessive speed. The record shows the effect of resonance under the main drivers, which oscillated roughly $1\frac{1}{2}$ in. above the normal surface of the rail and $\frac{7}{8}$ below it. In the majority of cases the wheel moves in phase with the unbalanced weight but in some instances it moved up when the unbalanced weight was down, indicating a speed in excess of resonance for the particular case.

REFERENCES

Den Hartog, J. P., *Mechanical Vibrations.* McGraw-Hill Book Co., Inc., New York, 1934.

Doherty, R. E., and Keller, E. G., *Mathematics of Modern Engineering.* John Wiley and Sons, Inc., New York, 1936.

Kimball, A. L., *Vibration Prevention in Engineering.* John Wiley and Sons, Inc., New York, 1932.

Marié, Georges, *Traité de Stabilité du Matérial des Chemins de Fer.* Librairie Ch. Béranger, Paris et Liége.

Timoshenko, S. P., *Vibration Problems in Engineering*, Second Edition. D. Van Nostrand Co., Inc., New York, 1937.

Cain, B. S., "Safe Operation of High Speed Locomotives." *A.S.M.E. Trans.*, 1935, p. 471.

Carter, F. W., "On the Action of a Locomotive Driving Wheel." *Proc. Roy. Soc.*, Series A, Vol. 112 (1926), p. 151.

Carter, F. W., "On the Stability of Running of Locomotives." *Proc. Roy. Soc.*, Series A, Vol. 121 (1928), p. 585.

Carter, F. W., "The Running of Locomotives with Reference to Their Tendency to Derail." Institution of Civil Engineers, *Selected Paper No. 91* (1930).

Dymond, A. W. J., "Some Factors Affecting the Riding of Coaching Stock." *Proc. Inst. Mech. Eng.*, 1931, p. 465.

Eksergian, R., "Stresses in Locomotive Frames." *A.S.M.E. Pamphlet*, 1922.

Eksergian, R., "The Balancing and Dynamic Rail Pressure of Locomotives." *A.S.M.E. Trans.* 1929 (2), p. 53.

Giesl-Gieslingen, A., "Riding Qualities of Passenger Cars." *Railway Age*, Vol. 101 (1936), p. 214.

Gouldthorpe, H. W., "Flexible Mounting of Diesel Engine Power Plants for Transportation Service." *Mech. Engineering*, March, 1938, p. 253.

HIRSHFELD, C. F., and PIRON, E. H., "Rubber Cushioning Devices." *A.S.M.E. Trans.* 1937, p. 471.

HULL, E. H., "The Use of Rubber in Vibration Isolation." *A.S.M.E. Trans.* 1937, p. A–109.

JACKLIN, H. M., "Human Reactions to Vibration." *S.A.E. Trans.* 1936, p. 401.

JACKLIN, H. M. and LIDDELL, G. J., "Riding Comfort Analysis." *Research Bulletin No. 44*, Engineering Experiment Station, Purdue University.

KEYS, W. C., "Rubber Springs," *Mech. Engineering*, Vol. 59, May, 1937, p. 345.

KINDL, C. H., "New Features in Shock Absorbers with Inertia Control." *S.A.E. Trans.* 1933, p. 172.

LANGER, B. F., and SHAMBERGER, J. P., "Lateral Oscillations of Rail Vehicles." *A.S.M.E. Trans.* 1935, p. 481.

OLLEY, MAURICE, "Independent Wheel Suspension—Its Whys and Wherefores." *S.A.E. Trans.* 1934, p. 73.

PATON, C. R., "Frame Design and Front End Stability." *S.A.E. Trans.* 1932, p. 305.

SMITH, J. F. D., "Rubber Mountings." *A.S.M.E. Trans.* 1938, p. A–13.

TAUB, A., "Resilient Mountings as Applied to Automotive Engines." *S.A.E. Trans.* 1934, p. 136.

WIDNEY, S. W., "Spring, Tire and Shock Absorber Testing Development." *S.A.E. Journal*, 1936, p. 99.

Some Physical Properties of Rubber. United States Rubber Products, Inc., Detroit, 1936.

"Track Tests of Electric Locomotives." *Railway Age*, Vol. 101 (1936), pp. 374 and 412.

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